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# Massachusetts Mathematics Curriculum Framework

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Department of Education

November 2000

# The Commonwealth of Massachusetts
# Department of Education

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*David P. Driscoll*
*Commissioner of Education*


November 2000

Dear Colleagues,

I am pleased to present to you the Massachusetts *Mathematics Curriculum Framework* that was adopted by the Board of Education in July 2000. This Framework presents the revised statewide guidelines for learning, teaching, and assessment in mathematics for the Commonwealth's public schools. Based on scholarship, sound research, and effective practice, the Framework will enable teachers and administrators to strengthen curriculum and instruction from prekindergarten through grade 12.

I am proud of the work that has been accomplished. The comments and suggestions received on the 1995 version of the *Mathematics Curriculum Framework*, as well as comments on working drafts of this version, have strengthened this Framework. I want to thank everyone who worked with us to create a high quality document that provides challenging learning standards for Massachusetts students.

We will continue to work with schools and districts in implementing the 2000 *Mathematics Curriculum Framework* over the next several years, and we encourage your comments as you use it. All of the curriculum frameworks are subject to continuous review and improvement, for the benefit of the students of the Commonwealth.

Thank you again for your ongoing support and for your commitment to achieving the goals of education reform.

Sincerely,

David P. Driscoll
Commissioner of Education

## Acknowledgments

The 2000 *Mathematics Curriculum Framework* is the result of the contributions of many mathematics educators across the state. Because of the broad-based, participatory nature of the revision process, this document cannot reflect all of the professional views of every contributor. It reflects instead a balanced synthesis of their suggestions. The Department of Education wishes to thank all of the groups that contributed to the development of these mathematics standards: the Mathematics Revision Panel, the Revision Steering Committee, the Mathematics/Science Advisory Council, grade-span teacher groups, professional educational associations and organizations, and all of the individual teachers, administrators, mathematicians, mathematics education faculty, and parents who took the time to provide thoughtful comments during the public comment period.

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# *Introduction*

*When we cannot use the compass of mathematics or the torch of experience . . . it is certain that we cannot take a single step forward.*

*– Voltaire*

The 2000 Mathematics Curriculum Framework is one of seven sets of standards created to advance educational reform in Massachusetts. It is the work of teachers and administrators in pre-kindergarten through grade 12, mathematics education professors, university mathematicians, and community members working with staff from the Department of Education.

## Organization of the document

The **guiding principles** articulate a set of beliefs about teaching, learning, and assessing mathematics in Massachusetts. Five **strands** organize the mathematical content: Number Sense and Operations; Patterns, Relations, and Algebra; Geometry; Measurement; and Data Analysis, Statistics, and Probability. At the beginning of each strand are the **broad concepts** from *NCTM Standards 2000*; they guide the clustering of the learning standards in that strand. In grades PreK–6, each of the broad concepts is identified by an icon (✚, ●, ▲, ■), and the icon at the end of each standard indicates the NCTM concept to which it is linked. Because the standards are grouped by broad concepts, the order in which they appear in each grade span is not necessarily the order in which they might be taught.

The **learning standards** are organized by two-year grade spans or for single-year courses. The grade spans are PreK–K, 1–2, 3–4, 5–6, 7–8, 9–10, and 11–12, and the courses are Algebra I, Geometry, Algebra II, and Precalculus. Each learning standard in each grade span or course has a unique identifier that consists of:

- the grade level (K, 2, 4, 6, 8, 10, or 12) or course (AI = Algebra I, G = Geometry, AII = Algebra II, and PC = Precalculus),
- the strand (N = Number Sense and Operations, P = Patterns, Relations, and Algebra, G = Geometry, M = Measurement, and D = Data Analysis, Statistics, and Probability), and
- the standard number.

For example, standard 4.G.6 is the sixth standard of the Geometry strand in the grades 3–4 grade span. Standard AI.P.12 is the twelfth standard of the Patterns, Relations, and Algebra strand in the Algebra I course. This numbering system allows teachers to also organize the standards by grade span. For example, sixth grade teachers preparing their curriculum can distinguish grades 5–6 standards in each strand by identifying all of the standards beginning with a 6.

The learning standards specify what students should know and be able to do as learners of mathematics at the end of each grade span or course. Students are held responsible for learning standards listed at earlier grade spans as well as their current grade span. Following the standards are **selected problems and activities** that clarify the standards and help teachers use them. Those marked with a † are drawn from *NCTM Standards 2000*. Each grade span also includes **exploratory concepts and skills** that are not intended to be assessed at that grade level. The exploratory concepts and skills usually appear as learning standards at a higher grade level and are assessed then.

Throughout this document, the standards are written to allow time for study of additional material at every grade level and for advanced courses in middle and high school. At the secondary school level, the order of the standards is not meant to imply a necessary sequence of topics. Not only is the sequence of topics flexible, but schools may also make choices about course organization and sequence. Schools may offer a sequence of courses emphasizing a particular mathematical subject or a sequence of multistrand courses. For schools choosing to offer courses that emphasize one subject, the document provides sets of standards for Algebra I, Geometry, Algebra II, and Precalculus. Schools may choose to offer Geometry before or after Algebra II. As the Department found in a survey of mathematics department chairs, teachers, and curriculum coordinators throughout the state, if students are to be able to take a calculus course in grade 12, they must usually take Algebra I in grade 8.[1] Indeed, a recent report from the U.S. Department of Education articulated the need to:

- Provide all students the opportunity to take Algebra I or a similarly demanding course that includes fundamental algebraic concepts in the 8th grade to enable students to take more advanced mathematics and science courses in all four years of high school.
- Build the groundwork for success in algebra by providing a rigorous curriculum in grades K–7 that moves beyond arithmetic and prepares students for the transition to algebra.
- Ensure that all students, parents, teachers, and counselors understand the importance of students' early study of algebra as well as continued study of rigorous mathematics and science in high school.[2]

To provide for more local control and flexibility in secondary school mathematics curricula, two assessment options at the end of grade 8 are being planned. One option will be based on the multistrand standards for grades 7 and 8. The other option will be an end-of-course assessment for Algebra I based on the standards listed for Algebra I.

Although the Department presents standards for only one post-Algebra II course—Precalculus—there are other mathematics courses that school districts might offer concurrent with or subsequent to Precalculus. Among these options are probability and statistics and discrete mathematics. Schools should also provide interested students with enrichment options in mathematics such as advanced placement courses, independent research, internships, or study of special topics.

## Development of the standards

The development of these standards is based upon two reform initiatives in Massachusetts, the Education Reform Act of 1993 and Partnerships Advancing the Learning of Mathematics and Science, the Statewide Systemic Initiative funded by the National Science Foundation since 1992. The initial set of standards was completed and approved by the Board of Education in 1995. Because the Education Reform Act required that all standards documents be reviewed and revised periodically, a mathematics revision panel was appointed in the summer of 1998. After reviewing the comments that the Department had received on the original set of mathematics standards, the revision panel examined these standards, assessed their appropriateness, and presented a draft to the Board of Education in September 1999.

The panel drew on the work of the National Council of Teachers of Mathematics (NCTM); reports of the Mathematical Association of America, the American Mathematical Society, and the American Association for the Advancement of Science; NCTM's *Principles and Standards for School Mathematics* October 1998 discussion draft; data from the Third International Mathematics and Science Study; the National Research Council's *National Science Education Standards*; and results from the initial administration of the Massachusetts Comprehensive Assessment System (MCAS).

Based on the comments that the Department received on this draft, further revisions were made. Drawing on the published version of *NCTM Standards 2000*, the final draft reflects the work of many PreK–12 mathematics educators, mathematicians, and others who helped clarify and refine the standards and ensure their developmental appropriateness.

# Guiding Philosophy

This curriculum framework envisions all students in the Commonwealth achieving mathematical competence through a strong mathematics program that emphasizes problem solving, communicating, reasoning and proof, making connections, and using representations. Acquiring such competence depends in large part on a clear, comprehensive, coherent, and developmentally appropriate set of standards to guide curriculum expectations.

## Problem Solving

Problem solving is both a means of developing students' knowledge of mathematics and a critical outcome of a good mathematics education. As such, it is an essential component of the curriculum. A mathematical problem, as distinct from an exercise, requires the solver to search for a method for solving the problem rather than following a set procedure. Mathematical problem solving, therefore, requires an understanding of relevant concepts, procedures, and strategies. To become good problem solvers, students need many opportunities to formulate questions, model problem situations in a variety of ways, generalize mathematical relationships, and solve problems in both mathematical and everyday contexts.

## Communicating

The ability to express mathematical ideas coherently to different audiences is an important skill in a technological society. Students develop this skill and deepen their understanding of mathematics when they use accurate mathematical language to talk and write about what they are doing. They clarify mathematical ideas as they discuss them with peers, and reflect on strategies and solutions. By talking and writing about mathematics, students learn how to make convincing arguments and to represent mathematical ideas verbally, pictorially, and symbolically.

## Reasoning and Proof

From the early grades on, students develop their reasoning skills by making and testing mathematical conjectures, drawing logical conclusions, and justifying their thinking in developmentally appropriate ways. As they advance through the grades, students' arguments become more sophisticated and they are able to construct formal proofs. By doing so, students learn what mathematical reasoning entails.

## Making Connections

Mathematics is not a collection of separate strands or standards. Rather, it is an integrated field of study. Students develop a perspective of the mathematics field as an integrated whole by understanding connections within and outside of the discipline. It is important for teachers to demonstrate the significance and relevance of the subject by encouraging students to explore the connections that exist within mathematics, with other disciplines, and between mathematics and students' own experiences.

## Representations

Mathematics involves using various types of representations for mathematical objects and actions, including numbers, shapes, operations, and relations. These representations can be numerals or diagrams, algebraic expressions or graphs, or matrices that model a method for solving a system of equations. Students must learn to use a repertoire of mathematical representations. When they can do so, they have a set of tools that significantly expands their capacity to think mathematically.

# Guiding Principles

## Guiding Principle I: Learning

**Mathematical ideas should be explored in ways that stimulate curiosity, create enjoyment of mathematics, and develop depth of understanding.**

Students need to understand mathematics deeply and use it effectively. To achieve mathematical understanding, students should be actively engaged in doing meaningful mathematics, discussing mathematical ideas, and applying mathematics in interesting, thought provoking situations. Student understanding is further developed through ongoing reflection about cognitively demanding and worthwhile tasks.

Tasks should be designed to challenge students in multiple ways. Short- and long-term investigations that connect procedures and skills with conceptual understanding are integral components of an effective mathematics program. Activities should build upon curiosity and prior knowledge, and enable students to solve progressively deeper, broader, and more sophisticated problems. Mathematical tasks reflecting sound and significant mathematics should generate active classroom talk, promote the development of conjectures, and lead to an understanding of the necessity for mathematical reasoning.

## Guiding Principle II: Teaching

**An effective mathematics program focuses on problem solving and requires teachers who have a deep knowledge of mathematics as a discipline.**

Mathematical problem solving is the hallmark of an effective mathematics program. Skill in mathematical problem solving requires practice with a variety of mathematical problems as well as a firm grasp of mathematical techniques and their underlying principles. Armed with this deeper knowledge, the student can then use mathematics in a flexible way to attack various problems and devise different ways of solving any particular problem. Mathematical problem solving calls for reflective thinking, persistence, learning from the ideas of others, and going back over one's own work with a critical eye. Success in solving mathematical problems helps to create an abiding interest in mathematics.

For a mathematics program to be effective, it must also be taught by knowledgeable teachers. According to Liping Ma, "The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics."[3]
A landmark study in 1996 found that students with initially comparable academic achievement levels had vastly different academic outcomes when teachers' knowledge of

the subject matter differed.[4] The message from the research is clear: having knowledgeable teachers really does matter; teacher expertise in a subject drives student achievement.

National data show that "nearly one-third of all secondary school teachers who teach mathematics have neither a major nor a minor in the subject itself, in mathematics education, or even in a related discipline." [5] While there are very effective teachers who do not have a major or minor in mathematics or in a related field, the goal should be that all future teachers have concentrated study in the field of mathematics. "Improving teachers' content subject matter knowledge and improving students' mathematics education are thus interwoven and interdependent processes that must occur simultaneously." [6]

## Guiding Principle III: Technology

**Technology is an essential tool in a mathematics education.**

Technology enhances the mathematics curriculum in many ways. Tools such as measuring instruments, manipulatives (such as base ten blocks and fraction pieces), scientific and graphing calculators, and computers with appropriate software, if properly used, contribute to a rich learning environment for developing and applying mathematical concepts. However, appropriate use of calculators is essential; calculators should not be used as a replacement for basic understanding and skills. Moreover, the fourth and sixth grade state assessments do not permit the use of a calculator. Elementary students should learn how to perform thoroughly the basic arithmetic operations independent of the use of a calculator.[7] Although the use of a graphing calculator can help middle and secondary students to visualize properties of functions and their graphs, graphing calculators should be used to enhance their understanding and skills rather than replace them.

Technology enables students to communicate ideas within the classroom or to search for information in external databases such as the Internet, an important supplement to a school's internal library resources. Technology can be especially helpful in assisting students with special needs in regular and special classrooms, at home, and in the community.

Technology changes what mathematics is to be learned and when and how it is learned. For example, currently available technology provides a dynamic approach to such mathematical concepts as functions, rates of change, geometry, and averages that was not possible in the past. Some mathematics becomes more important because technology requires it, some becomes less important because technology replaces it, and some becomes possible because technology allows it.

## Guiding Principle IV: Equity

**All students should have a high quality mathematics program.**

All Massachusetts students should have high quality mathematics programs that meet the goals and expectations of these standards and address students' individual interests and talents. The standards provide for a broad range of students, from those requiring tutorial support to those with talent in mathematics. To promote achievement of these standards, teachers should encourage classroom talk, reflection, use of multiple problem solving strategies, and a positive disposition toward mathematics. They should have high expectations for all students. At every level of the education system, teachers should act on the belief that every child should learn challenging mathematics. Teachers and guidance personnel should advise students and parents about why it is important to take advanced courses in mathematics and how this will prepare students for success in college and the workplace.

All students should have the benefit of quality instructional materials, good libraries, and adequate technology. Practice and enrichment should extend beyond the classroom. Tutorial sessions, mathematics clubs, competitions, and apprenticeships are examples of mathematics activities that promote learning. Because mathematics is the cornerstone of many disciplines, a comprehensive curriculum should include applications to everyday life and modeling activities that demonstrate the connections among disciplines. Schools should also provide opportunities for communicating with experts in applied fields to enhance students' knowledge of these connections.

## Guiding Principle V: Assessment

**Assessment of student learning in mathematics should take many forms to inform instruction and learning.**

A comprehensive assessment program is an integral component of an instructional program. It provides students with frequent feedback on their performance, teachers with diagnostic tools for gauging students' depth of understanding of mathematical concepts and skills, parents with information about their children's performance in the context of program goals, and administrators with a means for measuring student achievement.

Assessments take a variety of forms, require varying amounts of time, and address different aspects of student learning. Having students "think aloud" or talk through their solutions to problems permits identification of gaps in knowledge and errors in reasoning. By observing students as they work, teachers can gain insight into students' abilities to apply appropriate mathematical concepts and skills, make conjectures, and draw conclusions. Homework, mathematics journals, portfolios, oral performances, and group projects offer additional means for capturing students' thinking, knowledge of mathematics, facility with the language of mathematics, and ability to communicate what they know to others.

Tests and quizzes assess knowledge of mathematical facts, operations, concepts, and skills and their efficient application to problem solving. They can also pinpoint areas in need of more practice or teaching. Taken together, the results of these different forms of assessment provide rich profiles of students' achievements in mathematics and serve as the basis for identifying curricula and instructional approaches to best develop their talents.

Assessment should also be a major component of the learning process. As students help identify goals for lessons or investigations, they gain greater awareness of what they need to learn and how they will demonstrate that learning. Engaging students in this kind of goal-setting can help them reflect on their own work, understand the standards to which they are held accountable, and take ownership of their learning.

## Number Sense and Operations

*"Can you do addition?" the White Queen asked.*
*"What's one and one and one and one and one and one and one and one and one and one?"*
*"I don't know," said Alice. "I lost count."*
– *Lewis Carroll,* Through the Looking Glass

The study of numbers and operations is the cornerstone of the mathematics curriculum. Learning what numbers mean, how they may be represented, relationships among them, and computations with them is central to developing number sense.

Research in developmental psychology and in mathematics education has shown that young children have a great deal of informal knowledge of mathematics. As early as age three, children begin counting and quantifying, and demonstrate an eagerness to do so. Capitalizing on this informal knowledge and interest, education in the early years focuses on developing children's facility with oral counting and recognition of numerals and word names for numbers. Experience with counting naturally extends to quantification. Children count objects and learn that the sizes, shapes, positions, or purposes of objects do not affect the total number of objects in a group. One-to-one correspondence, with its matching of elements between two sets, provides the foundation for the comparison of groups. Combining and partitioning groups of objects set the stage for operations with whole numbers and the identification of equal parts of groups.

In the early elementary grades, students count and compute with whole numbers, learn different meanings of the operations and relationships among them, and apply the operations to the solutions of problems. "Knowing basic number combinations—the single-digit addition and multiplication pairs and their counterparts for subtraction and division—is essential. Equally essential is computational fluency—having and using efficient and accurate methods for computing." [8] Once teachers are confident that students understand the underlying structure of a particular operation, they should teach students the conventional algorithm for the operation. While students will not be asked to demonstrate use of standard algorithms on the grade four MCAS mathematics tests, they are expected to be introduced to them as theoretically and practically significant methods of computing.[9] After students have learned how to use the conventional algorithm for an operation, whatever they then choose to use on a routine basis should be judged on the basis of efficiency and accuracy. No matter what method students use, they should be able to explain their method, understand that many methods exist, and see the usefulness of methods that are efficient, accurate, and general.[10]

As they progress through the elementary grades, students compute with multi-digit numbers, estimate in order to verify results of computations with larger numbers, and use concrete objects to model operations with fractions, mixed numbers, and decimals. By the end of their elementary school years, students choose operations appropriately, estimate to solve problems mentally, and compute with whole numbers.

Mathematics in the middle school centers on understanding and computing with rational numbers, and on the study of ratio and proportion (what they are and how they are used to solve problems). Students achieve competence with rational number computations and the application of the order of operations rule to prepare for high school.

At the high school level, understanding systems of numbers is enhanced through exploration of real numbers and computations with them. Thereafter, students investigate complex numbers and relationships between the real and complex numbers. Students expand their knowledge of counting techniques, permutations, and combinations, and apply those techniques to the solution of problems.

As students develop competence with numbers and computation, they construct the scaffolding necessary to build an understanding of number systems. Students not only compute and solve problems with different types of numbers, but also explore the properties of operations on these numbers. Through investigation of relationships among whole numbers, integers, rational numbers, real numbers, and complex numbers, students gain a robust understanding of the structure of our number system.

Technology in the Number Sense and Operations strand is used to facilitate investigation of mathematical concepts, skills, and strategies. Calculators and computers enhance students' abilities to explore relationships among different sets of numbers (e.g., the relationship between fractions and decimals, fractions and percents, and decimals and percents), investigate alternative computational methods (e.g., generating the product of a pair of multi-digit numbers on a calculator when the multiplication key cannot be used), verify results of computations done with other tools, compute with very large and very small numbers in scientific notation form, and learn the rule for the order of operations.

## Patterns, Relations, and Algebra

> *I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours. I found it quite enthralling.*
> – *Agatha Christie*, An Autobiography

Algebra emerged through the analysis of solutions to equations, while the concept of a function developed as the insights and techniques of calculus began to spread. Patterns, relations, and algebra are integral elements in the study of mathematics. All students

should understand how patterns, relations, and functions are interrelated; be able to represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to understand quantitative relationships; and analyze change in various contexts.

The foundation for the study of patterns, relations, and algebra can be constructed in the PreK–K years and expanded gradually throughout the other years. All students should be aware of the mathematics in patterns and use mathematical representations to describe patterns. Young students can identify, translate, and extend repeating rhythmic, verbal, and visual patterns. They can recognize patterns and relationships among objects, and sort and classify them, observing similarities and differences. They can then probe more deeply into the study of patterns as they explore the properties of the operations of addition and multiplication.

Through numerous explorations, elementary grade students deepen their understanding of pattern and work informally with the concept of function. It is important that the concept of a variable is developed for them through practical situations, for example, as they engage in such basic activities as listing the cost of one pencil at 50¢, two pencils at ?, three pencils at ? , . . . n pencils at ?.

Investigating patterns helps older students understand the concept of constant growth as they analyze sequences like 1, 3, 5, 7, . . . . These students should contrast this type of change with other relationships as evidenced in sequences such as 1, 2, 4, 8, . . . ; 1, 3, 6, 10, 15, . . . (the sum of the first n positive integers); and 1, 1/10, 1/100, 1/1000, . . . . In middle and high school, students build on prior experiences as they compare sequences and functions represented in recursive and explicit forms.

As students advance through the grades, their work with patterns, functions, and algebra progresses in mathematical sophistication. They learn that change is a central idea in the study of mathematics and that multiple representations are needed to express change. They identify, represent, and analyze numerical relationships in tables, charts, and graphs. They learn about the importance and strength of proportional reasoning as a means of solving a variety of problems. While understanding linear functions and their graphs is a realistic goal for the middle school student, students deepen their study of functions in the secondary years. They engage in problems that feature additional types of polynomial functions, and rational, exponential, logarithmic, trigonometric, and other families of functions.

Graphing calculators and computer software with spreadsheet and graphics capabilities are ideal resources that help students make connections among different representations of the functions. The meaning and importance of domain, range, roots, optimum values, periodicity, and other terms come alive when experienced through technology. With appropriate instruction, students move readily among symbolic, numeric, and graphic representations of functions. Through insightful examples, secondary students learn that functions are a key concept with connections not only to calculus but also to transformational geometry and topics in discrete mathematics.

> *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles, and other geometrical figures, without which means it is humanly impossible to comprehend a single word.*
>
> – *Galileo Galilei,* Opere Il Saggiatore

Geometry, spatial sense, and measurement were among the sources of the earliest mathematical endeavors. Ancient people, in their efforts to manage their lands, conduct commerce, and describe natural forms and patterns, began to rely on abstract shapes and standard units of measurement to communicate with each other. *Euclidean geometry,* a landmark in the development of mathematics and other academic disciplines, is the study of points, lines, planes, and other geometric figures, using a modified version of the assumptions of Euclid, who is thought to have lived about 300 BC. The geometry in Euclid's *Elements of Geometry* was a logical system based on ten assumptions. Five of the assumptions were called common notions (axioms, or self-evident truths), and the other five were postulates (required conditions). The resulting logical system was taken as a model for deductive reasoning and profoundly influenced all branches of knowledge. Indeed, the development of the axiomatic approach to geometry extends to the present.[11]

Today, students have many of the same needs that their ancestors had thousands of years ago. They need to understand the structure of space and the spatial relations around them, measure many aspects of their environment, and communicate this structure, these relations, and their measurements to others. Instruction in geometry and measurement is designed to address this need.

Before students begin school, they have developed some knowledge of the physical and spatial world. They have explored size, shape, position, and orientation of objects in everyday activities at home, on the neighborhood playground, or at a supermarket. They have become familiar with two- and three-dimensional shapes. The language describing location and orientation of objects—words such as right, left, above, below, top, bottom, and between—are part of children's daily talk. However, everyday use of the language of measurement and geometry often differs from formal use in mathematics. These differences must be addressed through instruction. Students should be expected to use clear and increasingly precise language in their mathematical talk.

In the early grades, children explore shapes and the relationships among them that build on their natural understanding. As they progress through elementary school, students identify the components, attributes, and properties of different shapes, including sides, corners or vertices, edges, interiors, and exteriors. With time, they develop procedures to identify and categorize shapes by referring to their components, attributes, and properties. Students investigate these features dynamically by using mirrors, paper folding, hand drawing, and computer drawing. Still operating on concrete objects, students develop the idea of transformations by recognizing changes effected by slides, flips, and turns, not only on individual

objects but also on combinations of objects. Investigations of simple transformations lead to the concept of congruence.

In middle school and high school, students solve problems in other areas of mathematics using geometric concepts, including coordinate geometry, perspective drawings, and projections of three-dimensional objects. They use mechanical and electronic tools to construct common geometric shapes and patterns, and to develop the idea of geometric similarity, which can be integrated with the ideas of ratio and proportion. Students can also apply methods developed in the geometric context to make sense of fractions and variables, construct graphs and other representations of data, and make and interpret maps, blueprints, and schematic drawings.

In high school, students use formal reasoning to justify conclusions about geometry and its relationship to other areas of mathematics. They recognize the logical structure of the system of geometric axioms, become increasingly proficient in proving theorems within the axiomatic system, and use axioms and theorems to verify conjectures generated through their own work or by their peers. Students apply coordinate geometry to the solution of problems and extend transformational geometry to a variety of congruence and similarity transformations and their composition.

## MEASUREMENT

*Measure what is measurable, and make measurable what is not so.*
*– Galileo Galilei*

Measurement is best learned through direct applications or as part of other mathematical topics. A measurable attribute of an object is a characteristic that is most readily quantified and compared. Many attributes, such as length, perimeter, area, volume, and angle measure, come from the geometric realm. Other attributes are physical, such as temperature and mass. Still other attributes, such as density, are not readily measurable by direct means.

In PreK–K, students begin to make qualitative comparisons between physical objects, (e.g., which object is longer or shorter, which is lighter or heavier, which is warmer or colder), and begin to use nonstandard units of measurement for quantitative comparisons. Building on existing measurement ideas, students in grades 1 and 2 become competent with standard units of measurement. Students gain understanding of ratio and proportion in the middle grades, and apply their newfound knowledge to making scale drawings and maps that accurately reflect the dimensions of the landscape or the objects they represent. Greater familiarity with ratios enhances students' understanding of the derived attributes (speed, density, and trigonometric ratios), their applications, and the use of conversion factors to change a base unit in a measure.

At all levels, students develop respect for precision and accuracy by learning to select the tools and units of measurement appropriate to the situation. They also learn to analyze possible and real errors in their measurements and how those errors may be compounded in computations.

*Life is a school of probability.*
*– Walter Bagehot*

Education in a free society must prepare citizens to make informed choices in all areas of their lives. They must be able to grasp the information being presented, analyze it, and make reasoned decisions. To accomplish these goals, students learn to collect, organize, and display relevant data to answer questions that can be addressed with data; use appropriate statistical methods and predictions that are based on data; develop and evaluate inferences and predictions that are based on data; and apply basic concepts of probability.

In the early grades students learn how to collect data, observe patterns in the data, and organize and analyze the data to draw conclusions. To organize and display their data, they begin by using concrete and pictorial representations, and gradually learn to use tables, bar and line graphs, and data line plots. As students advance through the grades, they explore more complex forms of representation, including multiple-line graphs, circle graphs, and frequency tables.

In their study of data and statistics, students shift their perspective from viewing data as a set of individual pieces of information to an understanding of data as a coherent set with its own collective properties. This shift is emphasized in the middle grades when students study characteristics of sets of data, including measures of central tendency and techniques for displaying these characteristics, e.g., stem-and-leaf plots and scatterplots. Students learn how to select and construct representations most appropriate for the data and how to avoid misleading and inappropriate representations.

In high school, students gain insight into the use of trend lines and measures of spread for analyzing data. Students use technology to estimate and find lines of best fit for scatterplots. They categorize data by the type of model that best represents them, design surveys to generate data, and learn to choose representative samples and identify biases in the samples and survey questions.

Probability may be called the study of the laws of chance. In the elementary grades, students begin the study of probability by conducting experiments with spinners, counters, number cubes, and other concrete objects. They learn to record outcomes of individual experiments, and to organize and analyze results. They identify certain, possible, and impossible events.

In the middle grades, students enumerate all possible outcomes of simple experiments and determine probabilities to solve problems. Through the exploration of various problem situations, students learn to distinguish between independent and dependent events. By representing problem situations both numerically and geometrically, students can begin to develop an understanding of probabilities for simple compound events.

In high school, as they compare results of experiments with their theoretical predictions, students gain an understanding of the difference between predicted and actual outcomes. They apply counting techniques, use multiple representations to solve complex probability problems, and investigate probability distributions.

# Learning Standards by Strand for Grades PreK–6

- ✚ Understand numbers, ways of representing numbers, relationships among numbers, and number systems
- ● Understand meanings of operations and how they relate to one another
- ▲ Compute fluently and make reasonable estimates

## GRADES PreK–K LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

K.N.1 Count by ones to at least 20. ✚

K.N.2 Match quantities up to at least 10 with numerals and words. ✚

K.N.3 Identify positions of objects in sequences (e.g., first, second) up to fifth. ✚

K.N.4 Compare sets of up to at least 10 concrete objects using appropriate language (e.g., none, more than, fewer than, same number of, one more than) and order numbers. ✚

K.N.5 Understand the concepts of whole and half. ✚

K.N.6 Identify U.S. coins by name. ✚

K.N.7 Use objects and drawings to model and solve related addition and subtraction problems to ten. ●

K.N.8 Estimate the number of objects in a group and verify results. ▲

## Exploratory Concepts and Skills

- ✔ Count by ones, beginning from any number in the counting sequence.
- ✔ Represent quantities using concrete objects, and investigate the partitioning of sets. Identify equal parts of groups.
- ✔ Create problems that can be solved using addition and subtraction.

## Grades PreK–K Selected Problems or Classroom Activities

EXAMPLES

*Refers to standards K.N.1–K.N.5*

Listen to stories and explore books that incorporate number concepts.

*Refers to standards K.N.1, K.N.2, K.N.4, and K.N.8*

Have each child estimate the number of seeds in a slice of watermelon by inspection. Remove and count the seeds and compare the estimate to the count. Children then draw and color pictures of slices of watermelon, paste the seeds on their drawings, record the number of seeds, and compare their watermelon slices to tell who has more seeds.

*Refers to standards K.N.1 and K.N.3*

Engage in games, songs, nursery rhymes, and dances that incorporate number sequences.

## GRADES 1–2 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

2.N.1 Name and write (in numerals) whole numbers to 1000, identify the place values of the digits, and order the numbers. ✚

2.N.2 Identify and distinguish among multiple uses of numbers, including cardinal (to tell how many) and ordinal (to tell which one in an ordered list), and numbers as labels and as measurements. ✚

2.N.3 Identify and represent common fractions (½, ⅓, ¼) as parts of wholes, parts of groups, and numbers on the number line. ✚

2.N.4 Compare whole numbers using terms and symbols, e.g., less than, equal to, greater than (<, =, >). ✚

2.N.5 Identify odd and even numbers and determine whether a set of objects has an odd or even number of elements. ✚

2.N.6 Identify the value of all U.S. coins, and $1, $5, $10, and $20 bills. Find the value of a collection of coins and dollar bills and different ways to represent an amount of money up to $5. Use appropriate notation, e.g., 69¢, $1.35. ✚

2.N.7 Demonstrate an understanding of various meanings of addition and subtraction, e.g., addition as combination (plus, combined with, more); subtraction as comparison (how much less, how much more), equalizing (how many more are needed to make these equal), and separation (how much remaining). ●

2.N.8 Understand and use the inverse relationship between addition and subtraction (e.g., 8 + 6 = 14 is equivalent to 14 – 6 = 8 and is also equivalent to 14 – 8 = 6) to solve problems and check solutions. ●

2.N.9 Know addition facts (addends to ten) and related subtraction facts, and use them to solve problems. ▲

2.N.10 Demonstrate the ability to add and subtract three-digit numbers accurately and efficiently. ▲

2.N.11 Demonstrate in the classroom an understanding of and the ability to use the conventional algorithms for addition (two 3-digit numbers and three 2-digit numbers) and subtraction (two 3-digit numbers). ▲

2.N.12 Estimate, calculate, and solve problems involving addition and subtraction of two-digit numbers. Describe differences between estimates and actual calculations. ▲

## Exploratory Concepts and Skills

- ✔ Use concrete materials to investigate situations that lead to multiplication and division.
- ✔ Develop and use strategies for addition and subtraction of multi-digit whole numbers. Check by estimation.
- ✔ Investigate addition of common fractions, e.g., ½ + ½ = 1, ¼ + ¼ = ½.
- ✔ Understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.

*Refers to standard 2.N.1*

*Use 8, 6, and 4.*

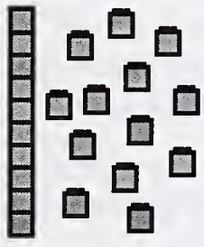
Write the smallest three-digit number: _____

Write the greatest three-digit number: _____

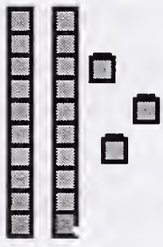
Write other numbers using the same digits: _____ _____ _____ _____

*Refers to standard 2.N.1†*

(a)
23 ones



(b)
1 ten and 13 ones



(c)
2 ten and 3 ones

*Refers to standard 2.N.3*

Count the small squares and color ¼ of them.

*Refers to standard 2.N.6*

P N N P N N . . .

P stands for penny and N stands for nickel. If the pattern continues until there are 12 coins altogether, what is the total value of all 12 coins?

## GRADES 3–4 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

4.N.1 Exhibit an understanding of the base ten number system by reading, modeling, writing, and interpreting whole numbers to at least 100,000; demonstrating an understanding of the values of the digits; and comparing and ordering the numbers. ✚

4.N.2 Represent, order, and compare large numbers (to at least 100,000) using various forms, including expanded notation, e.g., $853 = 8 \times 100 + 5 \times 10 + 3$. ✚

4.N.3 Demonstrate an understanding of fractions as parts of unit wholes, as parts of a collection, and as locations on the number line. ✚

4.N.4 Select, use, and explain models to relate common fractions and mixed numbers (½, ⅓, ¼, ⅕, ⅙, ⅛, 1/10, 1/12, and 1½), find equivalent fractions, mixed numbers, and decimals, and order fractions. ✚

4.N.5 Identify and generate equivalent forms of common decimals and fractions less than one whole (halves, quarters, fifths, and tenths). ✚

4.N.6 Exhibit an understanding of the base ten number system by reading, naming, and writing decimals between 0 and 1 up to the hundredths. ✚

4.N.7 Recognize classes (in particular, odds, evens; factors or multiples of a given number; and squares) to which a number may belong, and identify the numbers in those classes. Use these in the solution of problems. ✚

4.N.8 Select, use, and explain various meanings and models of multiplication and division of whole numbers. Understand and use the inverse relationship between the two operations. ●

4.N.9 Select, use, and explain the commutative, associative, and identity properties of operations on whole numbers in problem situations, e.g., $37 \times 46 = 46 \times 37$, $(5 \times 7) \times 2 = 5 \times (7 \times 2)$. ●

4.N.10 Select and use appropriate operations (addition, subtraction, multiplication, and division) to solve problems, including those involving money. ●

4.N.11 Know multiplication facts through $12 \times 12$ and related division facts. Use these facts to solve related multiplication problems and compute related problems, e.g., $3 \times 5$ is related to $30 \times 50$, $300 \times 5$, and $30 \times 500$. ▲

4.N.12 *Add and subtract (up to five-digit numbers) and multiply (up to three digits by two digits) accurately and efficiently. ▲

* Although this standard is appropriate as stated for this grade span, the state assessment program at the 3-4 grade span will test multiplication of only up to two digits by two digits at the present time.

## GRADES 3–4 LEARNING STANDARDS (CONTINUED)

4.N.13 Divide up to a three-digit whole number with a single-digit divisor (with or without remainders) accurately and efficiently. Interpret any remainders. ▲

4.N.14 Demonstrate in the classroom an understanding of and the ability to use the conventional algorithms for addition and subtraction (up to five-digit numbers), and multiplication (up to three digits by two digits). ▲

4.N.15 Demonstrate in the classroom an understanding of and the ability to use the conventional algorithm for division of up to a three-digit whole number with a single-digit divisor (with or without remainders). ▲

4.N.16 Round whole numbers through 100,000 to the nearest 10, 100, 1000, 10,000, and 100,000. ▲

4.N.17 Select and use a variety of strategies (e.g., front-end, rounding, and regrouping) to estimate quantities, measures, and the results of whole-number computations up to three-digit whole numbers and amounts of money to $1000, and to judge the reasonableness of the answer. ▲

4.N.18 Use concrete objects and visual models to add and subtract common fractions. ▲

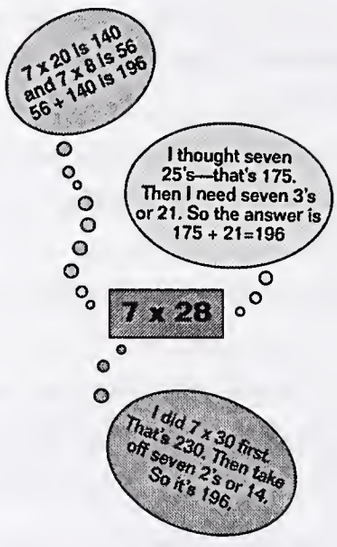
## Exploratory Concepts and Skills

- ✔ Extend multiplication and division to larger-digit numbers.
- ✔ Use models to explore multiplication and division with fractions (to twelfths) and decimals.
- ✔ Investigate number theory concepts, e.g., prime and composite numbers.
- ✔ Investigate the concept of ratio, e.g., the number of students to the number of teachers.
- ✔ Use concrete objects and visual models to add and subtract common decimals.
- ✔ Explore numbers less than zero by extending the number line and by using familiar applications such as temperature.
- ✔ Investigate the distributive property of multiplication over addition for single-digit multipliers, e.g., $7 \times (10 + 5)$ is equivalent to $7 \times 10 + 7 \times 5$.

## GRADES 3–4 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standards 4.N.8 and 4.N.11†*


7 x 20 is 140
and 7 x 8 is 56
56 + 140 is 196
I thought seven
25's—that's 175.
Then I need seven 3's
or 21. So the answer is
175 + 21=196
7 x 28
I did 7 x 30 first.
That's 230. Then take
off seven 2's or 14.
So it's 196.

*Refers to standards 4.N.10, 4.N.12, 4.N.16, and 4.N.17*

Hat — $4.52

Socks — $1.99

Sweater — $9.41

Scarf — $3.95

You have $20.
Estimate to find out if you have enough to buy all four items.
Explain how you made your estimate.

*Refers to standards 4.N.10 and 4.N.12*

A to D is 16 miles.

A to B is 7 miles.

B to C is 6 miles.

C to D is ____ miles.

## GRADES 5–6 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

6.N.1 Demonstrate an understanding of positive integer exponents, in particular, when used in powers of ten, e.g., $10^2$, $10^5$. ✚

6.N.2 Demonstrate an understanding of place value to billions and thousandths. ✚

6.N.3 Represent and compare very large (billions) and very small (thousandths) positive numbers in various forms such as expanded notation without exponents, e.g., $9724 = 9 \times 1000 + 7 \times 100 + 2 \times 10 + 4$. ✚

6.N.4 Demonstrate an understanding of fractions as a ratio of whole numbers, as parts of unit wholes, as parts of a collection, and as locations on the number line. ✚

6.N.5 Identify and determine common equivalent fractions, mixed numbers, decimals, and percents. ✚

6.N.6 Find and position integers, fractions, mixed numbers, and decimals (both positive and negative) on the number line. ✚

6.N.7 Compare and order integers (including negative integers), and positive fractions, mixed numbers, decimals, and percents. ✚

6.N.8 Apply number theory concepts—including prime and composite numbers, prime factorization, greatest common factor, least common multiple, and divisibility rules for 2, 3, 4, 5, 6, 9, and 10—to the solution of problems. ✚

6.N.9 Select and use appropriate operations to solve problems involving addition, subtraction, multiplication, division, and positive integer exponents with whole numbers, and with positive fractions, mixed numbers, decimals, and percents. ●

6.N.10 Use the number line to model addition and subtraction of integers, with the exception of subtracting negative integers. ●

6.N.11 Apply the Order of Operations for expressions involving addition, subtraction, multiplication, and division with grouping symbols (+, –, ×, ÷). ●

6.N.12 Demonstrate an understanding of the inverse relationship of addition and subtraction, and use that understanding to simplify computation and solve problems. ●

6.N.13 Accurately and efficiently add, subtract, multiply, and divide (with double-digit divisors) whole numbers and positive decimals. ▲

6.N.14 Accurately and efficiently add, subtract, multiply, and divide positive fractions and mixed numbers. Simplify fractions. ▲

## GRADES 5–6 LEARNING STANDARDS (CONTINUED)

6.N.15 Add and subtract integers, with the exception of subtracting negative integers. ▲

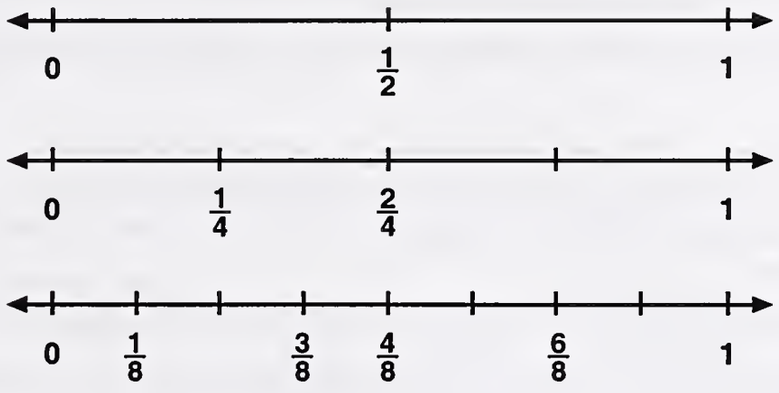
6.N.16 Estimate results of computations with whole numbers, and with positive fractions, mixed numbers, decimals, and percents. Describe reasonableness of estimates. ▲

### Exploratory Concepts and Skills

- ✔ Explore the addition and subtraction of positive and negative fractions.
- ✔ Investigate the concepts of ratio and proportion.
- ✔ Investigate the distributive property of multiplication over addition for double-digit multipliers, e.g., $12 \times (10 + 3)$ is equivalent to $12 \times 10 + 12 \times 3$.

*Refers to standards 6.N.4, 6.N.5, and 6.N.7†*

Fill in the missing fractions.




*Refers to standards 6.N.8*

Suppose that when a positive number is divided by 7, the result is $a$, and when the same number is divided by 8, the result is $b$.

How do $a$ and $b$ compare?

A. $a < b$

B. $a = b$

C $a > b$

D. It depends on the value of $n$.

- ✚ Understand patterns, relations, and functions
- ● Represent and analyze mathematical situations and structures using algebraic symbols
- ▲ Use mathematical models to represent and understand quantitative relationships
- ■ Analyze change in various contexts

## GRADES PREK–K LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

K.P.1 Identify the attributes of objects as a foundation for sorting and classifying, e.g., a red truck, a red block, and a red ball share the attribute of being red; a square block, a square cracker, and a square book share the attribute of being square shaped. ✚

K.P.2 Sort and classify objects by color, shape, size, number, and other properties. ✚

K.P.3 Identify, reproduce, describe, extend, and create color, rhythmic, shape, number, and letter repeating patterns with simple attributes, e.g., ABABAB . . . ✚

K.P.4 Count by fives and tens at least up to 50. ✚

## Exploratory Concepts and Skills

✔ Explore skip counting by twos.

## GRADES PREK–K SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standards K.P.1 and K.P.2*

Give children a group of blocks of varying sizes, shapes, and colors. Have the children put blocks together that are the same color and talk about the shapes and sizes of those blocks. Follow the same procedure for size and shape.

Play the game "Mystery Block." Give clues about the block and ask for the solution.

Example: The mystery block is

- Red
- Large
- Square

What is the mystery block?

*Refers to standard K.P.2*

Give each pair of students 25 assorted pattern blocks and have them sort the blocks. Ask students to compare the ways they sorted the blocks.

*Refers to standard K.P.3*

Recognize and predict word patterns in familiar stories and rhymes, e.g., Bill Martin's *Brown Bear, Brown Bear, What Do You See?*

*Refers to standard K.P.3*

Use unit blocks or beads to create and identify patterns of shape/color. Students could identify and replicate each other's patterns.

*Refers to standard K.P.3*

| ○ | □ | ↑ | ○ | □ | ↑ | ○ | — | — | — |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Ask the students to identify the pattern and draw shapes 8, 9, and 10.

## GRADES 1–2 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

2.P.1 Identify, reproduce, describe, extend, and create simple rhythmic, shape, size, number, color, and letter repeating patterns. ✚

2.P.2 Identify different patterns on the hundreds chart. ✚

2.P.3 Describe and create addition and subtraction number patterns, e.g., 1, 4, 7, 10 . . . ; or 25, 23, 21 . . . . ✚

2.P.4 Skip count by twos, fives, and tens up to at least 50, starting at any number. ✚

2.P.5 Construct and solve open sentences that have variables, e.g., □ + 7 = 10. ●

2.P.6 Write number sentences using +, –, <, = , and/or > to represent mathematical relationships in everyday situations. ▲

2.P.7 Describe functions related to trading, including coin trades and measurement trades, e.g., five pennies make one nickel or four cups make one quart. ▲

## Exploratory Concepts and Skills

✔ Investigate situations with variables as unknowns and as quantities that vary.

## GRADES 1–2 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standard 2.P.2*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

Ask the students what all of the grey numbers have in common.

*Refers to standard 2.P.5*

♣ + 4¢ = 9¢

♣ = ___¢

*Refers to standard 2.P.6*

How many pets do the Driscolls have if they have three turtles and four iguanas?

3 + 4 = 7 or 4 + 3 = 7.

*Refers to standard 2.P.6*

If Sandra has five marbles and Tom has three, who has more marbles?

Fill in the ☐ with <, = , or >.

**5 ☐ 3**

## GRADES 3–4 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

4.P.1 Create, describe, extend, and explain symbolic (geometric) and numeric patterns, including multiplication patterns like 3, 30, 300, 3000, . . . . ✚

4.P.2 Use symbol and letter variables (e.g., △, x) to represent unknowns or quantities that vary in expressions and in equations or inequalities (mathematical sentences that use =, <, >). ●

4.P.3 Determine values of variables in simple equations, e.g., 4106 – ∇ = 37; 5 = ❍ + 3 and □ – ❍ = 3. ●

4.P.4 Use pictures, models, tables, charts, graphs, words, number sentences, and mathematical notations to interpret mathematical relationships. ▲

4.P.5 Solve problems involving proportional relationships, including unit pricing (e.g., four apples cost 80¢, so one apple costs 20¢) and map interpretation (e.g., one inch represents five miles, so two inches represent ten miles). ▲

4.P.6 Determine how change in one variable relates to a change in a second variable, e.g., input-output tables. ■

## Exploratory Concepts and Skills

✔ Use concrete materials to build an understanding of equality and inequality.

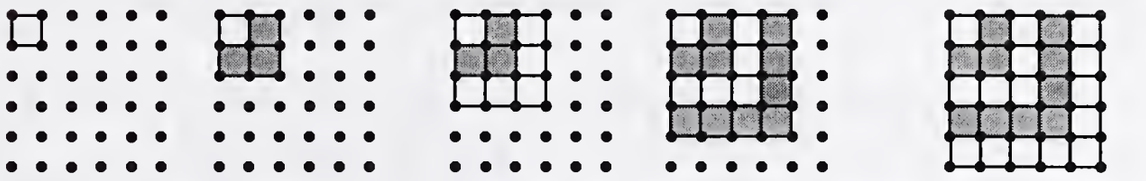
✔ Explore properties of equality in number sentences: when equals are added to equals, then the sums are equal; when equals are multiplied by equals, then the products are equal, e.g., if □ = 5, then $3 \times \square = 3 \times 5$.

## GRADES 3–4 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standard 4.P.1†*

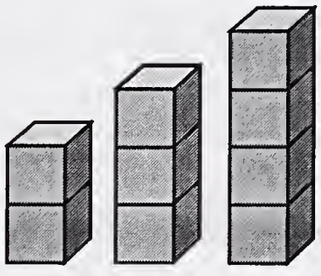
Use dot paper to extend the "growing squares" to 64 squares (adapted from Burton et al. 1992, p. 6).



1 1+3=4 1+3+5=9 1+3+5+7=16 1+3+5+7+9=25

*Refers to standards 4.P.1, 4.P.4, and 4.P.6†*

How many squares make up the surface of each tower of cubes (including the top and bottom)? As the tower gets taller, how does this number change?



| Number of cubes (N) | Number of squares on the surface |
|---|---|
| 1 | 6 |
| 2 | 10 |
| 3 | 14 |
| 4 | 18 |

A table used in the "tower of cubes" problem

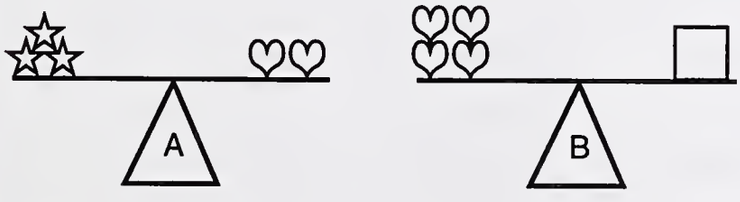
*Refers to standards 4.P.3 and 4.P.4*

♣ + 4¢ = 9¢ ♣ =___¢

♣ + ♠ = 8¢ ♠ =___¢

*Refers to standards 4.P.4 and 4.P.5*

Use the picture below to answer the questions.




1. How many stars will balance two squares?

2. How do you know?

## GRADES 5–6 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

6.P.1 Analyze and determine the rules for extending symbolic, arithmetic, and geometric patterns and progressions, e.g., ABBCCC; 1, 5, 9, 13 . . . ; 3, 9, 27, . . . ✚

6.P.2 Replace variables with given values and evaluate/simplify, e.g., 2(❍) + 3 when ❍ = 4. ●

6.P.3 Use the properties of equality to solve problems, e.g., if □ + 7 = 13, then □ = 13 – 7, therefore □ = 6; if 3 × □ = 15, then ⅓ × 3 × □ = ⅓ × 15, therefore □ = 5. ●

6.P.4 Represent real situations and mathematical relationships with concrete models, tables, graphs, and rules in words and with symbols, e.g., input-output tables. ▲

6.P.5 Solve linear equations using concrete models, tables, graphs, and paper-pencil methods. ▲

6.P.6 Produce and interpret graphs that represent the relationship between two variables in everyday situations. ▲

6.P.7 Identify and describe relationships between two variables with a constant rate of change. Contrast these with relationships where the rate of change is not constant. ■

## Exploratory Concepts and Skills

✔ Use physical models to investigate and describe how a change in one variable affects a second variable.

✔ Use models to develop understanding of slope as constant rate of change.

✔ Model situations with proportional relationships and solve problems.

## GRADES 5–6 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

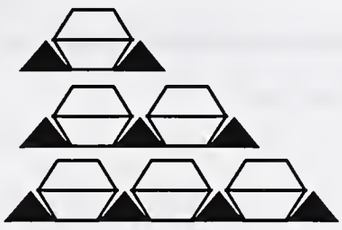
*Refers to standard 6.P.1*

Triangles and trapezoids were used to make a pattern.

Level 1:

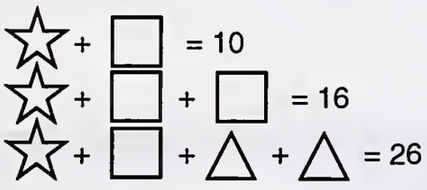
Level 2:

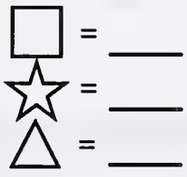
Level 3:



1. If the pattern above continues, how many black triangles are needed to build level 10? Explain how you know you are correct.
2. If the pattern above continues, how many white trapezoids are needed to build level 10? Explain how you know you are correct.

*Refers to standards 6.P.2, 6.P.3, and 6.P.5*



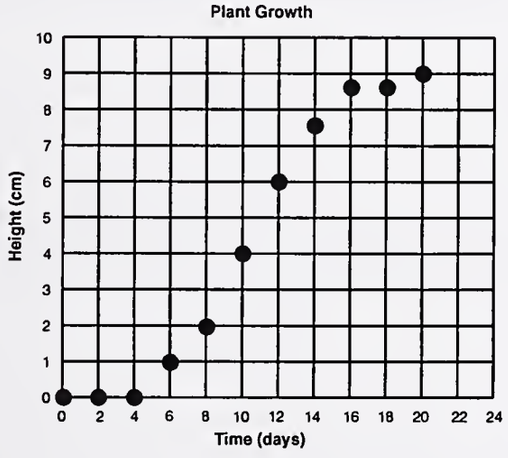




Explain your solution strategy.

*Refers to standards 6.P.4 and 6.P.6†*

Based on the data in the table, create a graph that shows the plant's growth over time.

| Time (days) | Height (cm) | Change (cm) |
|---|---|---|
| 0 | 0 | |
| 2 | 0 | 0 |
| 4 | 0 | 0 |
| 6 | 1 | 1 |
| 8 | 2 | 1 |
| 10 | 4 | 2 |
| 12 | 6 | 2 |
| 14 | 7.5 | 1.5 |
| 16 | 8.5 | 1 |
| 18 | 8.5 | 0 |
| 20 | 9 | 0.5 |



- ✚ Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- ● Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- ▲ Apply transformations and use symmetry to analyze mathematical situations
- ■ Use visualization, spatial reasoning, and geometric modeling to solve problems

## Grades PreK–K Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

K.G.1 Name, describe, sort, and draw simple two-dimensional shapes. ✚

K.G.2 Describe attributes of two-dimensional shapes, e.g., number of sides, number of corners. ✚

K.G.3 Name and compare three-dimensional shapes. ✚

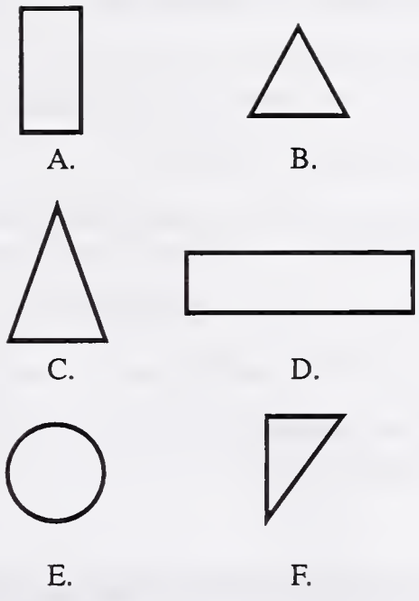
K.G.4 Identify positions of objects in space, and use appropriate language (e.g., beside, inside, next to, close to, above, below, apart) to describe and compare their relative positions. ●

## Exploratory Concepts and Skills

✔ Investigate symmetry of two- and three-dimensional shapes and constructions.

*Refers to standards K.G.1 and K.G.2*

1. Count the sides of each shape.
2. Count the corners of each shape.
3. Tell the name of each shape.




*Refers to standards K.G.1 and K.G.2*

Working with shapes: feel and identify shapes without looking, trace around templates of various shapes, eat snacks cut into various shapes, cut cookie dough into basic shapes.

*Refers to standards K.G.1 and K.G.2*

Ask students to find basic shapes in their environment, e.g., go on a "shape walk" indoors or out to find examples of circles, triangles, squares; have students bring in examples from home.

*Refers to standards K.G.4*

Have students explore positions in space (up, down, over, under, high, low, behind, inside, outside, on top) and relative distance between objects or locations (near, far, next to, apart, together).

## GRADES 1–2 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

2.G.1 Describe attributes and parts of two- and three-dimensional shapes, e.g., length of sides, and number of corners, edges, faces, and sides. ✚

2.G.2 Identify, describe, draw, and compare two-dimensional shapes, including both polygonal (up to six sides) and curved figures such as circles. ✚

2.G.3 Recognize congruent shapes. ✚

2.G.4 Identify shapes that have been rotated (turned), reflected (flipped), translated (slid), and enlarged. Describe direction of translations, e.g., left, right, up, down. ▲

2.G.5 Identify symmetry in two-dimensional shapes. ▲

2.G.6 Predict the results of putting shapes together and taking them apart. ■

2.G.7 Relate geometric ideas to numbers, e.g., seeing rows in an array as a model of repeated addition. ■

## Exploratory Concepts and Skills

- ✔ Investigate symmetry in two-dimensional shapes with mirrors or by paper folding.
- ✔ Explore intersecting, parallel, and perpendicular lines.
- ✔ Create mental images of geometric shapes using spatial memory and spatial visualization.
- ✔ Recognize and represent shapes from different perspectives.
- ✔ Recognize geometric shapes and structures in the environment and specify their location.
- ✔ Identify relative positions, e.g., closer, farther, higher, lower, etc.
- ✔ Find and name locations on maps and express simple relationships, e.g., near to, far away from.

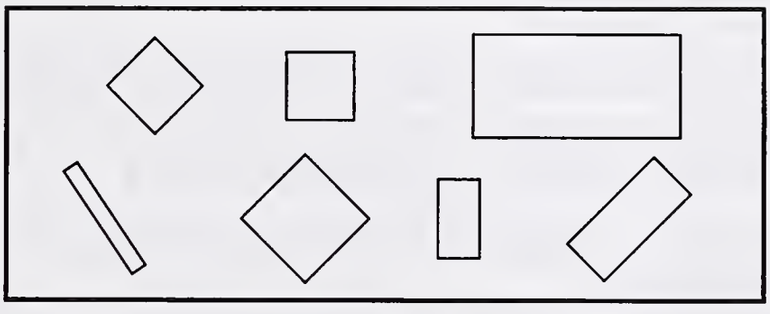
*Refers to standards 2.G.1, 2.G.2, and 2.G.3*

Use geoblocks. For each block:

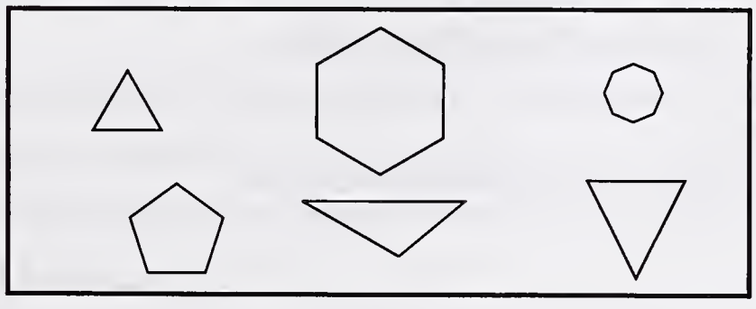
- Trace the faces.
- Identify the shape of each face.
- Tell the number of congruent faces.

*Refers to standards 2.G.1 and 2.G.2*

Use the sets below to answer the question.



I



II

What rule did Ray use to sort the shapes above into two groups?

## GRADES 3–4 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

4.G.1 Compare and analyze attributes and other features (e.g., number of sides, faces, corners, right angles, diagonals, and symmetry) of two- and three-dimensional geometric shapes. ✚

4.G.2 Describe, model, draw, compare, and classify two- and three-dimensional shapes, e.g., circles, polygons—especially triangles and quadrilaterals—cubes, spheres, and pyramids. ✚

4.G.3 Recognize similar figures. ✚

4.G.4 Identify angles as acute, right, or obtuse. ✚

4.G.5 Describe and draw intersecting, parallel, and perpendicular lines. ✚

4.G.6 Using ordered pairs of numbers and/or letters, graph, locate, identify points, and describe paths (first quadrant). ●

4.G.7 Describe and apply techniques such as reflections (flips), rotations (turns), and translations (slides) for determining if two shapes are congruent. ▲

4.G.8 Identify and describe line symmetry in two-dimensional shapes. ▲

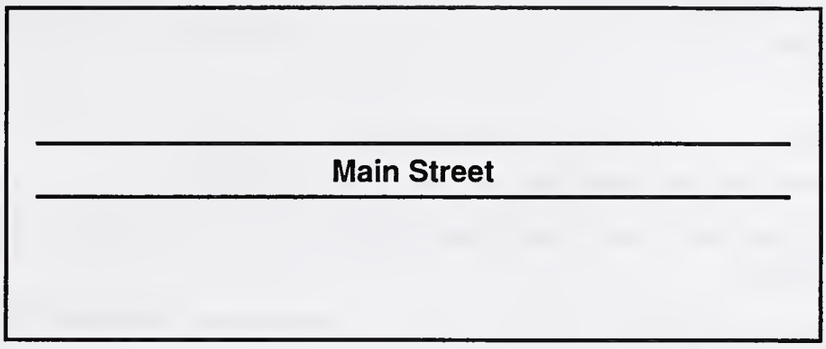
4.G.9 Predict and validate the results of partitioning, folding, and combining two- and three-dimensional shapes. ▲

## Exploratory Concepts and Skills

✔ Predict and describe results of transformations (e.g., translations, rotations, and reflections) on two-dimensional shapes.

✔ Investigate two-dimensional representations of three-dimensional objects.

*Refers to standards 4.G.4 and 4.G.5*



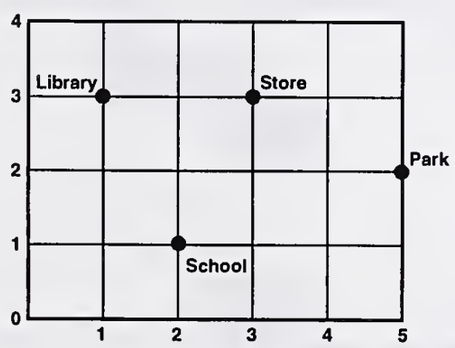

Definitions: Obtuse angle—an angle greater than a right angle; acute angle—an angle less than a right angle.

1. Draw Broadway Street *parallel* to Main Street. Write the name Broadway on this street.
2. Draw Birch Street *perpendicular* to Main Street. Write the name Birch on this street.
3. Draw Walnut Street so that it *intersects* Main Street but is not perpendicular to Main Street. Write the name Walnut on this street.
4. Determine what types of angles are present.

*Refers to standard 4.G.6†*

Use the map below to answer the following questions.

1. What are the coordinates of the school?
2. Which building is at 3,3? 5,2?
3. Moving along the grid lines, how many blocks is it from the library to the park?



## GRADES 5–6 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

6.G.1 Identify polygons based on their properties, including types of interior angles, perpendicular or parallel sides, and congruence of sides, e.g., squares, rectangles, rhombuses, parallelograms, trapezoids, and isosceles, equilateral, and right triangles. ✚

6.G.2 Identify three-dimensional shapes (e.g., cubes, prisms, spheres, cones, and pyramids) based on their properties, such as edges and faces. ✚

6.G.3 Identify relationships among points, lines, and planes, e.g., intersecting, parallel, perpendicular. ✚

6.G.4 *Graph points and identify coordinates of points on the Cartesian coordinate plane (all four quadrants). ●

6.G.5 Find the distance between two points on horizontal or vertical number lines. ●

6.G.6 Predict, describe, and perform transformations on two-dimensional shapes, e.g., translations, rotations, and reflections. ▲

6.G.7 Identify types of symmetry, including line and rotational. ▲

6.G.8 Determine if two shapes are congruent by measuring sides or a combination of sides and angles, as necessary; or by motions or series of motions, e.g., translations, rotations, and reflections. ▲

6.G.9 Match three-dimensional objects and their two-dimensional representations, e.g., nets, projections, and perspective drawings. ■

## Exploratory Concepts and Skills

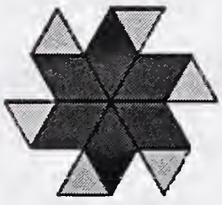
- ✔ Use manipulatives and technology to model geometric shapes.
- ✔ Investigate tessellations (tilings).
- ✔ Explore the angles formed by intersecting lines.
- ✔ Identify and draw shapes and figures from different views/perspectives.
- ✔ Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life.

*Although this standard is important and appropriate for this grade span, it will not be included in the state assessment program at the 5–6 grade span at the present time.

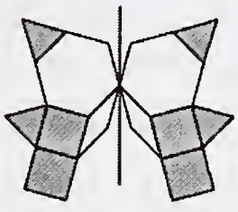
*Refers to standard 6.G.1†*

Draw several right triangles with two sides of equal length.

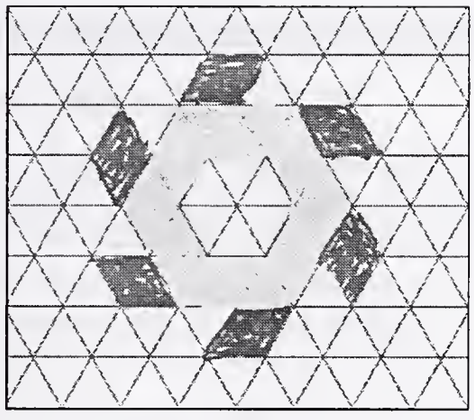
*Refers to standard 6.G.7†*



Rotational symmetry



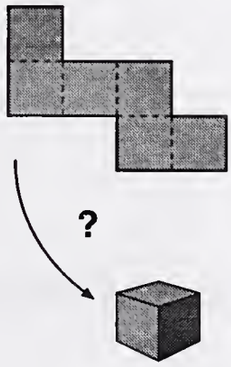
Line symmetry



What kind(s) of symmetry exist(s) in the figure above?

*Refers to standard 6.G.9†*

Is it possible to make the cube by folding the two-dimensional shape illustrated?



- ✚ Understand measurable attributes of objects and the units, systems, and processes of measurement
- ● Apply appropriate techniques, tools, and formulas to determine measurements

## GRADES PREK–K LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

K.M.1 Recognize and compare the attributes of length, volume/capacity, weight, area, and time using appropriate language, e.g., longer, taller, shorter, same length; heavier, lighter, same weight; holds more, holds less, holds the same amount. ✚

K.M.2 Make and use estimates of measurements from everyday experiences. ●

K.M.3 Use nonstandard units to measure length, area, weight, and capacity. ●

## Exploratory Concepts and Skills

- ✔ Explore and use standard units to measure and compare temperature, length, and time.
- ✔ Identify positions of events over time, e.g., earlier, later.

*Refers to standards K.M.1 and K.M.3*

Students can use concrete materials such as pieces of ribbon or string, popsicle sticks, or body parts to measure the height of their block constructions, the length or width of classroom materials, or the distance between objects.

*Refers to standards K.M.1 and K.M.3*

Use various sizes and shapes of containers to measure and mix sand, water, and mud, and compare the weights/capacities of various size containers.

*Refers to standards K.M.2 and K.M.3*

Estimate 1) how many cups of water are needed to fill a large container, 2) which container will hold the most jelly beans, or 3) the number of beads needed to make a necklace or bracelet. Count or measure to verify the actual results of the students' estimates.

*Refers to standard K.M.3*

Have students use paper clips to measure crayons, pencils, and rulers. Compare the lengths of these objects to each other and use appropriate vocabulary to describe them.

## GRADES 1–2 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

2.M.1 Identify parts of the day (e.g., morning, afternoon, evening), days of the week, and months of the year. Identify dates using a calendar. ✚

2.M.2 Tell time at quarter-hour intervals on analog and digital clocks using a.m. and p.m.✚

2.M.3 Compare the length, weight, area, and volume of two or more objects by using direct comparison.✚

2.M.4 Measure and compare common objects using metric and English units of length measurement, e.g., centimeter, inch. ✚

2.M.5 Select and correctly use the appropriate measurement tools, e.g., ruler, balance scale, thermometer. ✚

2.M.6 Make and use estimates of measurement, including time, volume, weight, and area. ●

## Exploratory Concepts and Skills

✔ Explore measurable attributes of objects, including length, perimeter, weight, area, volume, and temperature. Compare concrete objects using these measures.

## GRADES 1–2 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standards 2.M.1 and 2.M.2*

With the children's help, make a schedule of activities for the morning, recording times to the hour and half-hour. Set the alarms of both a digital clock and an analog clock to ring at the start of each new activity. Call on children to read the clocks to verify that they match the times in the schedules.

*Refers to standards 2.M.3, 2.M.4, and 2.M.5†*

A teacher had given her class a list of things to measure; because she was interested in finding out how the students would approach the task, she had left the choice of measuring tools up to them. Aliza was using a ruler when the teacher stopped by the desk to observe her measuring her book. "It's twelve inches," Aliza said as she wrote the measurement on the recording sheet. Next she measured her pencil, which was noticeably shorter than the book. The teacher observed that Aliza's hand slipped as she was aligning her ruler with the pencil. Aliza made no comment but recorded this measurement as twelve inches also.

"I notice that you wrote that each of these is twelve inches," said the teacher. "I'm confused. The book looks much longer than the pencil to me. What do you think?"

Aliza pushed both items close together and studied them. "You're right," she said. "The book is longer, but they are both twelve inches."

In her anecdotal records, the teacher noted what happened in order to address the issue in future lessons and conversations with Aliza and the class.

## GRADES 3–4 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

4.M.1 Demonstrate an understanding of such attributes as length, area, weight, and volume, and select the appropriate type of unit for measuring each attribute. ✚

4.M.2 Carry out simple unit conversions within a system of measurement, e.g., hours to minutes, cents to dollars, yards to feet or inches, etc. ✚

4.M.3 Identify time to the minute on analog and digital clocks using a.m. and p.m. Compute elapsed time using a clock (e.g., hours and minutes since...) and using a calendar (e.g., days since...). ●

4.M.4 Estimate and find area and perimeter of a rectangle, triangle, or irregular shape using diagrams, models, and grids or by measuring. ●

4.M.5 Identify and use appropriate metric and English units and tools (e.g., ruler, angle ruler, graduated cylinder, thermometer) to estimate, measure, and solve problems involving length, area, volume, weight, time, angle size, and temperature. ●

## Exploratory Concepts and Skills

✔ Develop the concepts of area and perimeter by investigating areas and perimeters of regular and irregular shapes created on dot paper, coordinate grids, or geoboards.

✔ Use concrete objects to explore volumes and surface areas of rectangular prisms.

✔ Investigate the use of protractors to measure angles.

✔ Identify common measurements of turns, e.g., 360° in one full turn, 180° in a half turn, and 90° in a quarter turn.

✔ Investigate areas of right triangles.

✔ Understand that measurements are approximations and investigate how differences in units affect precision.

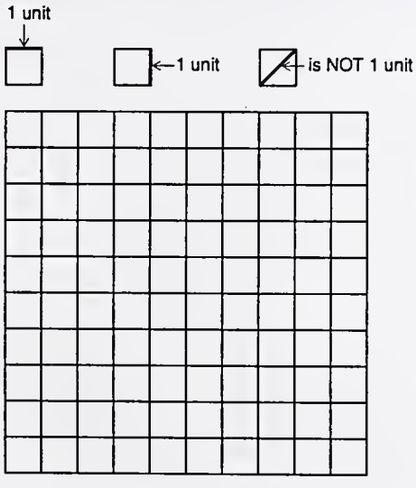
*Refers to standards 4.M.1 and 4.M.4*

Each ☐ is one square centimeter. What is the area of the shaded letter?

*Refers to standards 4.M.1 and 4.M.4*

The City Park Committee wants to fence in an area of the neighborhood park for young children to use as a playground. It has 12 sections of fence. Each section of fence is one unit long.

1. On a grid (example shown below), draw 4 different closed shapes using all 12 sections of fence.
2. What is the area of each of your shapes? Write the area inside each shape.
3. Decide which shape would be the best one for the playground. Circle your choice. Explain why this is the best shape for a playground.



## GRADES 5–6 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

6.M.1 Apply the concepts of perimeter and area to the solution of problems. Apply formulas where appropriate. ●

6.M.2 Identify, measure, describe, classify, and construct various angles, triangles, and quadrilaterals. ●

6.M.3 Solve problems involving proportional relationships and units of measurement, e.g., same system unit conversions, scale models, maps, and speed. ●

6.M.4 Find areas of triangles and parallelograms. Recognize that shapes with the same number of sides but different appearances can have the same area. Develop strategies to find the area of more complex shapes. ●

6.M.5 Identify, measure, and describe circles and the relationships of the radius, diameter, circumference, and area (e.g., d = 2r, p = C/d), and use the concepts to solve problems. ●

6.M.6 Find volumes and surface areas of rectangular prisms. ●

6.M.7 Find the sum of the angles in simple polygons (up to eight sides) with and without measuring the angles. ●

## Exploratory Concepts and Skills

✔ Explore various models for finding the area of a triangle, parallelogram, and trapezoid, and develop strategies for more complex shapes.

✔ Investigate volumes and surface areas of a variety of three-dimensional objects.

✔ Explore volume and surface areas of rectangular prisms, cylinders, and spheres.

*Refers to standard 6.M.4†*

Show how to find the area of the isosceles trapezoid by decomposing and rearranging it into a rectangle with the same area.

*Refers to standard 6.M.6*

Storage boxes are cube shaped and measure 4 inches on an edge. How many of these storage boxes are needed for 300 small cubes, 2 inches on an edge?

*Refers to standard 6.M.7*

Determine the sum of the measures of the angles of an equilateral triangle, a square, a regular pentagon, a regular hexagon, and a regular octagon.

- ✚ Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them
- ● Select and use appropriate statistical methods to analyze data
- ▲ Develop and evaluate inferences and predictions that are based on data
- ■ Understand and apply basic concepts of probability

## GRADES PREK–K LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

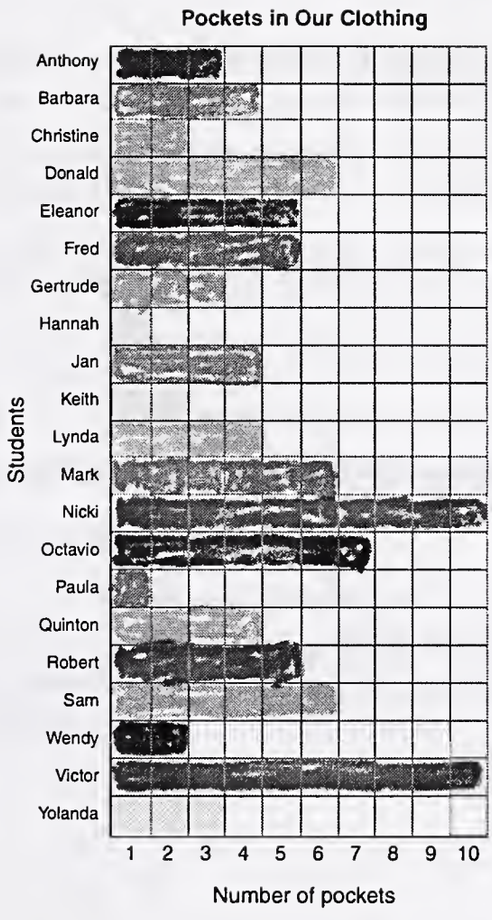
K.D.1 Collect, sort, organize, and draw conclusions about data using concrete objects, pictures, numbers, and graphs. ●

### Exploratory Concepts and Skills

✔ Collect and organize data in lists, tables, and simple graphs.

*Refers to standard K.D.1†*

Create a bar graph that illustrates the number of pockets in classmates' clothes.




*Refers to standard K.D.1*

Children use chips to represent objects in the stories below in order to be able to count and compare the objects more easily.

Story 1

Ms. Chapeau has large floppy hats. Put down one chip for each hat.

Ms. Chapeau has a red hat with stars.

She has a pink hat with flowers.

She has a green hat with ribbons.

She has a blue hat with feathers.

How many large floppy hats does she have?

Who has more hats?

Story 2

Mr. Miller has animal hats. Put down one chip for each hat.

Mr. Miller has a zebra hat with stripes.

He has a snake hat with scales.

He has a dog hat that barks.

How many animal hats does he have?

## GRADES 1–2 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

2.D.1 Use interviews, surveys, and observations to gather data about themselves and their surroundings. ✚

2.D.2 Organize, classify, represent, and interpret data using tallies, charts, tables, bar graphs, pictographs, and Venn diagrams; interpret the representations. ●

2.D.3 Formulate inferences (draw conclusions) and make educated guesses (conjectures) about a situation based on information gained from data. ▲

2.D.4 Decide which outcomes of experiments are most likely. ■

## Exploratory Concepts and Skills

✔ Investigate more likely, likely, and impossible outcomes by conducting experiments using spinners, counters, and other concrete objects.

✔ List and count the number of possible pairings of objects from two sets.

## GRADES 1–2 SELECTED PROBLEMS OR CLASSROOM ACTIVITIES

EXAMPLES

*Refers to standards 2.D.1 and 2.D.2*

Mr. Greenleaf's class collected data on the number of children in each student's family. The first graders found the numbers of children to be:

3 2 1 1 1 2 2 2 1 3 4 1 1 2 3 5 2 4

1. Make a tally sheet and then a bar graph to show the number of children in the students' families.
2. Describe the results, e.g., what is the most common or frequent number of children in a family? How many more students have families with 2 or fewer children compared with families with 3 or more children?

*Refers to standards 2.D.2 and 2.D.3*

Students each trace one of their own shoes on paper and cut it out. Next, they place their cutouts on a floor graph (pictograph) that identifies the kind of footwear each is wearing, e.g., sneakers, boots, sandals. Discuss what the graph shows. Extend the activity by using tally marks and making a bar graph on inch-square graph paper.

*Refers to standards 2.D.2 and 2.D.3*

José has lots of pennies, nickels, and dimes in his pocket. He takes out three coins and puts them on the table. How much money could be on the table? Make a list.

*Refers to standards 2.D.2 and 2.D.3*

Place one green apple and one red apple into a bag. Have the students predict which color apple will come out most often. Next, have each student in turn pick one apple from the bag without looking. Make a tally of red vs. green apples for the entire class's picks. Have the class compare their predictions with the outcome of the experiment.

## GRADES 3–4 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

4.D.1 Collect and organize data using observations, measurements, surveys, or experiments, and identify appropriate ways to display the data. ✚

4.D.2 Match representations of a data set such as lists, tables, or graphs (including circle graphs) with the actual set of data. ●

4.D.3 Construct, draw conclusions, and make predictions from various representations of data sets, including tables, bar graphs, pictographs, line graphs, line plots, and tallies. ▲

4.D.4 Represent the possible outcomes for a simple probability situation, e.g., the probability of drawing a red marble from a bag containing three red marbles and four green marbles. ■

4.D.5 List and count the number of possible combinations of objects from three sets, e.g., how many different outfits can one make from a set of three shirts, a set of two skirts, and a set of two hats? ■

4.D.6 Classify outcomes as certain, likely, unlikely, or impossible by designing and conducting experiments using concrete objects such as counters, number cubes, spinners, or coins. ■

## Exploratory Concepts and Skills

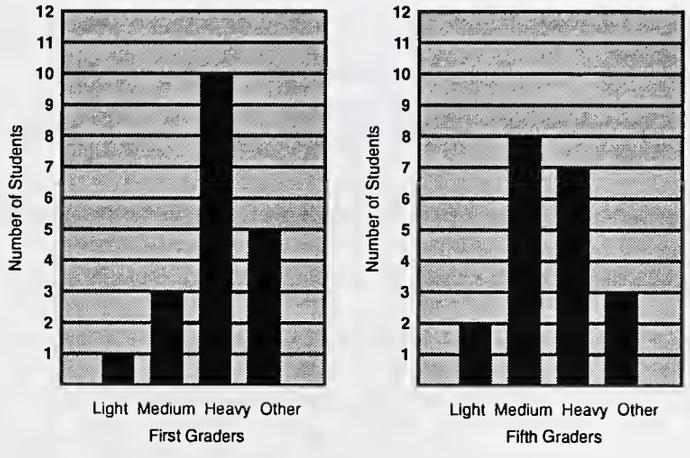
- ✔ Explore the concepts of median, mode, maximum and minimum, and range.
- ✔ Discuss what data-collection methods are appropriate for various types of investigations.
- ✔ Explore situations that involve probabilities of equally likely events.
- ✔ Investigate the construction of simple circle graphs.

*Refers to standards 4.D.1 and 4.D.3†*

During a presidential campaign, various news organizations are publishing polls that try to predict the outcome of the election. Mr. Schmid's fourth grade class talks about how such polls are conducted. "Is it practical to ask every voter in the United States?" the teacher asks. No, it would not be practical, the children agree. Conner suggests polling all the voters in our small town. Sabina does not think this is a good idea—we live on the East Coast, and perhaps voters on the West Coast have quite different political opinions. The class discusses how they might go about finding an appropriate sample of voters.

*Refers to standards 4.D.1 and 4.D.3†*

Gather data about the sleeping habits of students in at least two different grades in your school. Be sure to define your terms. Develop an appropriate way to display the data and discuss conclusions drawn from it.




| Type of Sleeper | |
|---|---|
| Light | Wakes up to the slightest noise |
| Medium | Wakes up to louder noises |
| Heavy | Sleeps through the night without waking up |
| Other | None of the previous three |

*Refers to standards 4.D.4 and 4.D.6*

There are two decks of cards.

Deck 1 has only clubs, spades, and diamonds, the same number of each. If you pick one card without looking, is it more likely that you will get a red card or a black card? Explain.

Deck 2 has only clubs and diamonds, the same number of each. If you pick one without looking, is it more likely that you will get a red card or a black card? Explain.

## GRADES 5–6 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

6.D.1 Describe and compare data sets using the concepts of median, mean, mode, maximum and minimum, and range. ●

6.D.2 Construct and interpret stem-and-leaf plots, line plots, and circle graphs.●

6.D.3 Use tree diagrams and other models (e.g., lists and tables) to represent possible or actual outcomes of trials. Analyze the outcomes. ■

6.D.4 Predict the probability of outcomes of simple experiments (e.g., tossing a coin, rolling a die) and test the predictions. Use appropriate ratios between 0 and 1 to represent the probability of the outcome and associate the probability with the likelihood of the event. ■

## Exploratory Concepts and Skills

- ✔ Set up and analyze capture-recapture experiments.
- ✔ Generate and group data, record the data using frequency tables and interpret the tables.
- ✔ Select, create, and use appropriate graphical representations of data, including histograms, box plots, and scatter plots.
- ✔ Compare different representations of the same data and evaluate how well each representation shows important aspects of the data.

*Refers to standard 6.D.1*

So far this term, Heidi has these scores on quizzes.

87, 86, 96, 87

What is the lowest score she can get on the one remaining quiz to have a final average (mean) score of 90?

A. 94

B. 97

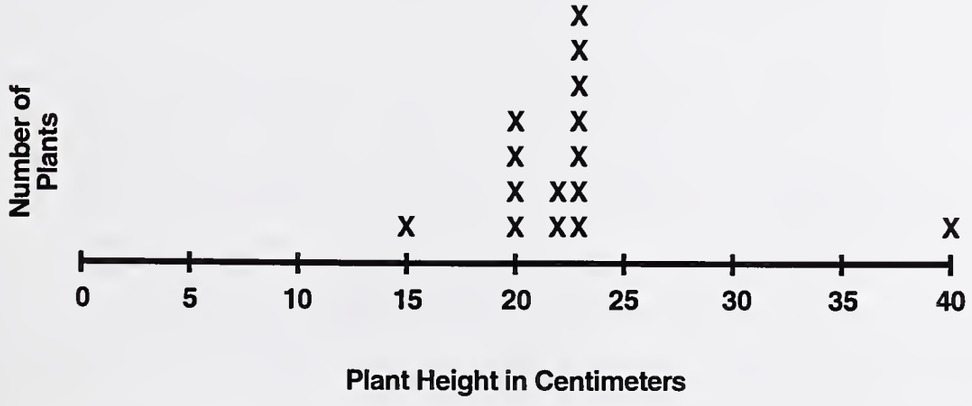
C. 90

D. 91

*Refers to standard 6.D.2†*

Create the line plot that corresponds to the information in the following tally chart.

| Plant Height Data | |
|---|---|
| Height (in cm) | Number of Plants |
| 15 | \| |
| 20 | \|\|/\ |
| 22 | \|\| |
| 23 | ~~\|\|\|\|~~ \|\| |
| 40 | \| |



*Refers to standard 6.D.4*

Jenna has a bag of saltwater taffy of which 8 are lemon, 6 are peppermint, and 10 are licorice. She offers the bag to Antonia, who takes one piece of taffy without looking. What are the chances that it is peppermint?

*Refers to standard 6.D.4*

Of the 640 students in a school, 428 were born in Massachusetts. If a newspaper reporter interviews one student at random, which is the best estimate of the probability that the student was born in Massachusetts

A. $\frac{1}{2}$

B. $\frac{2}{3}$

C. $\frac{2}{1}$

D. $\frac{3}{2}$

# Learning Standards by Grade Span or Course for Grades 7–12

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

## GRADES 7–8 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

8.N.1 Compare, order, estimate, and translate among integers, fractions and mixed numbers (i.e., rational numbers), decimals, and percents.

8.N.2 Define, compare, order, and apply frequently used irrational numbers, such as $\sqrt{2}$ and $\pi$.

8.N.3 Use ratios and proportions in the solution of problems, in particular, problems involving unit rates, scale factors, and rate of change.

8.N.4 Represent numbers in scientific notation, and use them in calculations and problem situations.

8.N.5 Apply number theory concepts, including prime factorization and relatively prime numbers, to the solution of problems.

8.N.6 Demonstrate an understanding of absolute value, e.g., $|-3| = |3| = 3$.

8.N.7 Apply the rules of powers and roots to the solution of problems. Extend the Order of Operations to include positive integer exponents and square roots.

8.N.8 Demonstrate an understanding of the properties of arithmetic operations on rational numbers. Use the associative, commutative, and distributive properties; properties of the identity and inverse elements (e.g., $-7 + 7 = 0$; $\frac{3}{4} \times \frac{4}{3} = 1$); and the notion of closure of a subset of the rational numbers under an operation (e.g., the set of odd integers is closed under multiplication but not under addition).

8.N.9 Use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems, e.g. multiplying by ½ or 0.5 is the same as dividing by 2.

8.N.10 Estimate and compute with fractions (including simplification of fractions), integers, decimals, and percents (including those greater than 100 and less than 1).

8.N.11 Determine when an estimate rather than an exact answer is appropriate and apply in problem situations.

8.N.12 Select and use appropriate operations—addition, subtraction, multiplication, division, and positive integer exponents—to solve problems with rational numbers (including negatives).

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

## GRADES 7–8 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

8.P.1 Extend, represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic expressions. Include arithmetic and geometric progressions, e.g., compounding.

8.P.2 Evaluate simple algebraic expressions for given variable values, e.g., $3a^2 - b$ for $a = 3$ and $b = 7$.

8.P.3 Demonstrate an understanding of the identity $(-x)(-y) = xy$. Use this identity to simplify algebraic expressions, e.g., $(-2)(-x+2) = 2x - 4$.

8.P.4 Create and use symbolic expressions and relate them to verbal, tabular, and graphical representations.

8.P.5 Identify the slope of a line as a measure of its steepness and as a constant rate of change from its table of values, equation, or graph. Apply the concept of slope to the solution of problems.

8.P.6 Identify the roles of variables within an equation, e.g., $y = mx + b$, expressing y as a function of x with parameters m and b.

8.P.7 Set up and solve linear equations and inequalities with one or two variables, using algebraic methods, models, and/or graphs.

8.P.8 Explain and analyze—both quantitatively and qualitatively, using pictures, graphs, charts, or equations—how a change in one variable results in a change in another variable in functional relationships, e.g., $C = \pi d$, $A = \pi r^2$ (A as a function of r), $A_{rectangle} = lw$ ($A_{rectangle}$ as a function of l and w).

8.P.9 Use linear equations to model and analyze problems involving proportional relationships. Use technology as appropriate.

8.P.10 Use tables and graphs to represent and compare linear growth patterns. In particular, compare rates of change and x- and y-intercepts of different linear patterns.

# GEOMETRY

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

## GRADES 7–8 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

8.G.1 Analyze, apply, and explain the relationship between the number of sides and the sums of the interior and exterior angle measures of polygons.

8.G.2 Classify figures in terms of congruence and similarity, and apply these relationships to the solution of problems.

8.G.3 Demonstrate an understanding of the relationships of angles formed by intersecting lines, including parallel lines cut by a transversal.

8.G.4 Demonstrate an understanding of the Pythagorean theorem. Apply the theorem to the solution of problems.

8.G.5 Use a straight-edge, compass, or other tools to formulate and test conjectures, and to draw geometric figures.

8.G.6 Predict the results of transformations on unmarked or coordinate planes and draw the transformed figure, e.g., predict how tessellations transform under translations, reflections, and rotations.

8.G.7 Identify three-dimensional figures (e.g., prisms, pyramids) by their physical appearance, distinguishing attributes, and spatial relationships such as parallel faces.

8.G.8 Recognize and draw two-dimensional representations of three-dimensional objects, e.g., nets, projections, and perspective drawings.

Understand measurable attributes of objects and the units, systems, and processes of measurement

Apply appropriate techniques, tools, and formulas to determine measurements

## GRADES 7–8 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

8.M.1 Select, convert (within the same system of measurement), and use appropriate units of measurement or scale.

8.M.2 Given the formulas, convert from one system of measurement to another. Use technology as appropriate.

8.M.3 Demonstrate an understanding of the concepts and apply formulas and procedures for determining measures, including those of area and perimeter/circumference of parallelograms, trapezoids, and circles. Given the formulas, determine the surface area and volume of rectangular prisms, cylinders, and spheres. Use technology as appropriate.

8.M.4 Use ratio and proportion (including scale factors) in the solution of problems, including problems involving similar plane figures and indirect measurement.

8.M.5 Use models, graphs, and formulas to solve simple problems involving rates, e.g., velocity and density.

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

## GRADES 7–8 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

8.D.1 Describe the characteristics and limitations of a data sample. Identify different ways of selecting a sample, e.g., convenience sampling, responses to a survey, random sampling.

8.D.2 Select, create, interpret, and utilize various tabular and graphical representations of data, e.g., circle graphs, Venn diagrams, scatterplots, stem-and-leaf plots, box-and-whisker plots, histograms, tables, and charts. Differentiate between continuous and discrete data and ways to represent them.

8.D.3 Find, describe, and interpret appropriate measures of central tendency (mean, median, and mode) and spread (range) that represent a set of data. Use these notions to compare different sets of data.

8.D.4 Use tree diagrams, tables, organized lists, basic combinatorics ("fundamental counting principle"), and area models to compute probabilities for simple compound events, e.g., multiple coin tosses or rolls of dice.

## Exploratory Concepts and Skills for Grades 7–8

### Number Sense and Operations

- ✔ Investigate the meaning of significant digits.
- ✔ Investigate negative integral exponents and their use in scientific and calculator notation.

### Patterns, Relations, and Algebra

- ✔ Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative and recursive (e.g., Pascal's triangle), and linear functional relationships.
- ✔ Use tables, graphs, and appropriate technology to explore quadratic and exponential growth patterns.
- ✔ Investigate the use of systems of equations, tables, and graphs to represent mathematical relationships.
- ✔ Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.

### Geometry

- ✔ Formulate and test conjectures about shapes that tessellate.
- ✔ Investigate trigonometric ratios in right triangles.
- ✔ Investigate right triangle relationships, such as those in 45°–45°–90° and 30°–60°–90° triangles.
- ✔ Explore proofs of the Pythagorean theorem.

### Measurement

- ✔ Given the formula, find surface area and volume of pyramids and cones.
- ✔ Select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision.
- ✔ Investigate formulas to determine the circumference and area of circles, and the perimeter and area of polygons.

### Data Analysis, Statistics, and Probability

- ✔ Investigate the notion of fairness in games.
- ✔ Make predictions, conduct experiments, and discuss discrepancies to develop understanding of actual versus predicted outcomes.
- ✔ Conduct repetitive experiments (e.g., repeated throwing of three or more dice) and compare the outcomes to predicted probabilities.

*Refers to standards 8.N.1, 8.N.10, and 8.N.12*

You purchase one dollar of stock in Global Enterprises, Inc. On day 1, it rises 50% in value. On day 2, it falls 50% in value. On day 3, it rises 50% in value. On day 4, it falls 50% in value.

How much (to the nearest penny) is it worth at the end of day 4?

*Refers to standard 8.N.5*

If M is an odd number, then which of the following statements are true?

1. 3M is an odd number.

2. $M^2$ is an odd number.

3. $(M + 3)^2$ is an odd number.

*Refers to standard 8.N.10*

If X% of 12.5 is 37.5, then the percent is greater than 100.
If X% of 12.5 is 6.25, then the percent is less than 100.

1. Explain why the percent is greater or less than 100.

2. Find X if X% of 62 is 186.

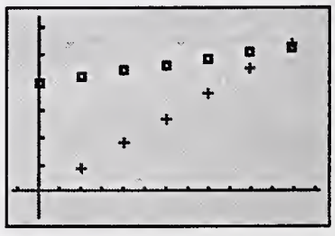
3. Find X if X% of 62 is 15.5.

*Refers to standards 8.P.9 and 8.P.10†*

Students can compare the charges for two telephone companies by making a table (a) and by representing the charges on a graphing calculator (b).

| No. of minutes | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| Keep-in-Touch | $20.00 | $21.00 | $22.00 | $23.00 | $24.00 | $25.00 | $26.00 |
| ChitChat | $0.00 | $4.50 | $9.00 | $13.50 | $18.00 | $22.50 | $27.00 |

(a)



(b)

*Refers to standards 8.P.6 and 8.P.8†*

Working with variables and equations is an important part of the middle-grades curriculum. Students' understanding of variable should go far beyond simply recognizing that letters can be used to stand for unknown numbers in equations (Schoenfeld and Arcavi 1988). The following equations illustrate several uses of variable encountered in the middle grades:

$$27 = 4x + 3$$

$$1 = t(1/t)$$

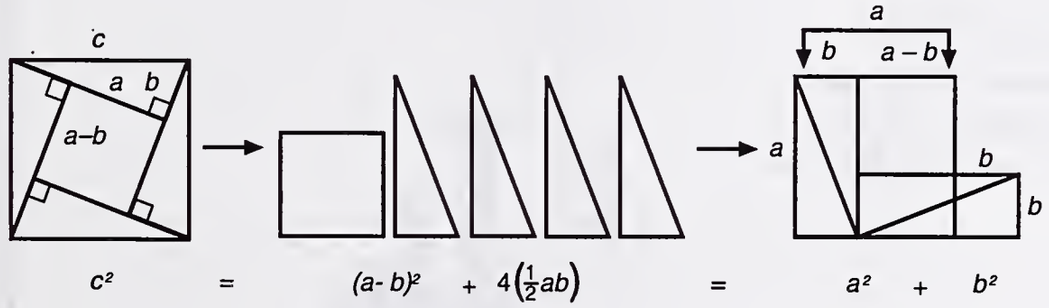
$$A = LW$$

$$y = 3x$$

The role of variable as "place holder" is illustrated in the first equation: $x$ is simply taking the place of a specific number that can be found by solving the equation. The use of variable in denoting a generalized arithmetic pattern is shown in the second equation; it represents an identity when $t$ takes on any real value except 0. The third equation is a formula, with $A$, $L$, and $W$ representing the area, length, and width, respectively, of a rectangle. The third and fourth equations offer examples of covariation: in the fourth equation, as $x$ takes on different values, $y$ also varies.

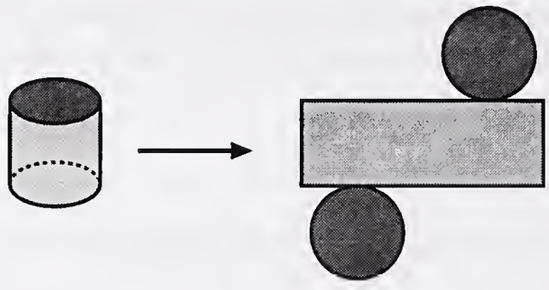
*Refers to standard 8.G.4†*

An algebraic explanation of a visual proof of the Pythagorean theorem.



*Refers to standard 8.M.3†*

Determine the surface area of a cylinder by determining the area of its net.



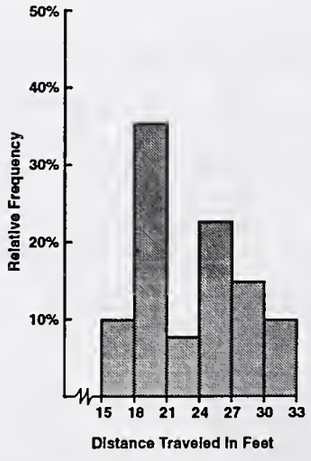
*Refers to standards 8.M.1 and 8.M.4*

At the end of every second mile of the Boston Marathon, a typical marathon runner takes a four ounce cup of water. Instead of drinking all of the water, the runner sips some of it and then throws the rest on his or her head or body to cool off.

1. Assuming the typical runner drinks half of the water in the cup, how many ounces of water would an average runner drink during an entire 26.2 mile marathon? Explain how you found your answer.
2. Suppose that all of the runners in the Boston Marathon behaved like the "typical" marathon runner described above. About how many gallons of water would the 40,700 runners in the 1996 Boston Marathon have used? Record each step that you used to find your answer.

*Refers to standard 8.D.2†*

A relative-frequency histogram for data for the distance a paper airplane with one paper clip travels.

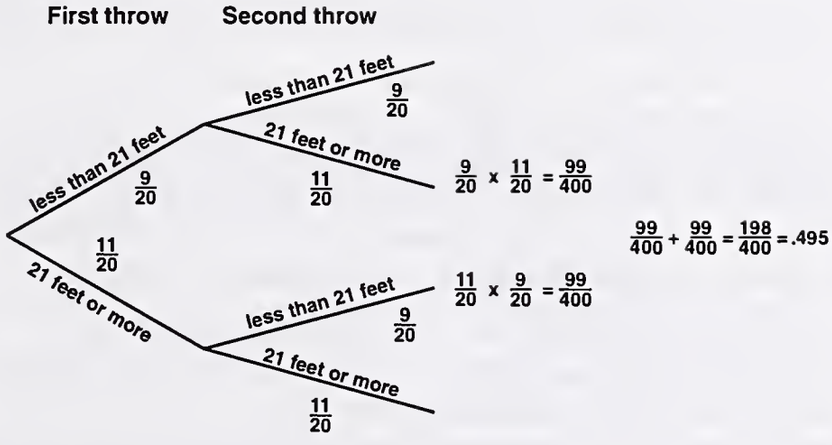



*Refers to standard 8.D.3*

Boaz took five tests, each worth 100 points. His average score was 85. What is the lowest score he could have received on one of the tests? Explain your answer.

*Refers to standard 8.D.4†*

A tree diagram for determining the probability of a compound event, given simple data.




*Refers to standard 8.D.4*

The sample space for the roll of two tetrahedral dice.

| Second die \ First die | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | 6 | 7 |
| 4 | 5 | 6 | 7 | 8 |

## Number Sense and Operations

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

### Grades 9–10 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

10.N.1 Identify and use the properties of operations on real numbers, including the associative, commutative, and distributive properties; the existence of the identity and inverse elements for addition and multiplication; the existence of $n^{th}$ roots of positive real numbers for any positive integer n; and the inverse relationship between taking the $n^{th}$ root of and the $n^{th}$ power of a positive real number.

10.N.2 Simplify numerical expressions, including those involving positive integer exponents or the absolute value, e.g., $3(2^4 - 1) = 45$, $4|3 - 5| + 6 = 14$; apply such simplifications in the solution of problems.

10.N.3 Find the approximate value for solutions to problems involving square roots and cube roots without the use of a calculator, e.g., $\sqrt{3^2 - 1} \approx 2.8$.

10.N.4 Use estimation to judge the reasonableness of results of computations and of solutions to problems involving real numbers.

## Patterns, Relations, and Algebra

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

### Grades 9–10 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

10.P.1 Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative, recursive (e.g., Fibonacci Numbers), linear, quadratic, and exponential functional relationships.

## Grades 9–10 Learning Standards (continued)

10.P.2 Demonstrate an understanding of the relationship between various representations of a line. Determine a line's slope and x- and y-intercepts from its graph or from a linear equation that represents the line. Find a linear equation describing a line from a graph or a geometric description of the line, e.g., by using the "point-slope" or "slope y-intercept" formulas. Explain the significance of a positive, negative, zero, or undefined slope.

10.P.3 Add, subtract, and multiply polynomials. Divide polynomials by monomials.

10.P.4 Demonstrate facility in symbolic manipulation of polynomial and rational expressions by rearranging and collecting terms; factoring (e.g., $a^2 - b^2 = (a + b)(a - b)$, $x^2 + 10x + 21 = (x + 3)(x + 7)$, $5x^4 + 10x^3 - 5x^2 = 5x^2(x^2 + 2x - 1)$); identifying and canceling common factors in rational expressions; and applying the properties of positive integer exponents.

10.P.5 Find solutions to quadratic equations (with real roots) by factoring, completing the square, or using the quadratic formula. Demonstrate an understanding of the equivalence of the methods.

10.P.6 Solve equations and inequalities including those involving absolute value of linear expressions (e.g., $|x - 2| > 5$) and apply to the solution of problems.

10.P.7 Solve everyday problems that can be modeled using linear, reciprocal, quadratic, or exponential functions. Apply appropriate tabular, graphical, or symbolic methods to the solution. Include compound interest, and direct and inverse variation problems. Use technology when appropriate.

10.P.8 Solve everyday problems that can be modeled using systems of linear equations or inequalities. Apply algebraic and graphical methods to the solution. Use technology when appropriate. Include mixture, rate, and work problems.

# Geometry

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

## Grades 9–10 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

10.G.1 Identify figures using properties of sides, angles, and diagonals. Identify the figures' type(s) of symmetry.

## Grades 9–10 Learning Standards (continued)

10.G.2 Draw congruent and similar figures using a compass, straightedge, protractor, and other tools such as computer software. Make conjectures about methods of construction. Justify the conjectures by logical arguments.

10.G.3 Recognize and solve problems involving angles formed by transversals of coplanar lines. Identify and determine the measure of central and inscribed angles and their associated minor and major arcs. Recognize and solve problems associated with radii, chords, and arcs within or on the same circle.

10.G.4 Apply congruence and similarity correspondences (e.g., $\triangle ABC \cong \triangle XYZ$) and properties of the figures to find missing parts of geometric figures, and provide logical justification.

10.G.5 Solve simple triangle problems using the triangle angle sum property and/or the Pythagorean theorem.

10.G.6 Use the properties of special triangles (e.g., isosceles, equilateral, 30°–60°–90°, 45°–45°–90°) to solve problems.

10.G.7 Using rectangular coordinates, calculate midpoints of segments, slopes of lines and segments, and distances between two points, and apply the results to the solutions of problems.

10.G.8 Find linear equations that represent lines either perpendicular or parallel to a given line and through a point, e.g., by using the "point-slope" form of the equation.

10.G.9 Draw the results, and interpret transformations on figures in the coordinate plane, e.g., translations, reflections, rotations, scale factors, and the results of successive transformations. Apply transformations to the solutions of problems.

10.G.10 Demonstrate the ability to visualize solid objects and recognize their projections and cross sections.

10.G.11 Use vertex-edge graphs to model and solve problems.

# Measurement

Understand measurable attributes of objects and the units, systems, and processes of measurement

Apply appropriate techniques, tools, and formulas to determine measurements

## Grades 9–10 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

10.M.1 Calculate perimeter, circumference, and area of common geometric figures such as parallelograms, trapezoids, circles, and triangles.

## Grades 9–10 Learning Standards (continued)

10.M.2 Given the formula, find the lateral area, surface area, and volume of prisms, pyramids, spheres, cylinders, and cones, e.g., find the volume of a sphere with a specified surface area.

10.M.3 Relate changes in the measurement of one attribute of an object to changes in other attributes, e.g., how changing the radius or height of a cylinder affects its surface area or volume.

10.M.4 Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements.

# Data Analysis, Statistics, and Probability

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

## Grades 9–10 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

10.D.1 Select, create, and interpret an appropriate graphical representation (e.g., scatterplot, table, stem-and-leaf plots, box-and-whisker plots, circle graph, line graph, and line plot) for a set of data and use appropriate statistics (e.g., mean, median, range, and mode) to communicate information about the data. Use these notions to compare different sets of data.

10.D.2 Approximate a line of best fit (trend line) given a set of data (e.g., scatterplot). Use technology when appropriate.

10.D.3 Describe and explain how the relative sizes of a sample and the population affect the validity of predictions from a set of data.

## Exploratory Concepts and Skills for Grades 9–10 and Single-Subject Courses

### Number Sense and Operations

- ✔ Analyze relationships among the various subsets of the real numbers (whole numbers, integers, rationals, and irrationals).
- ✔ Explore higher powers and roots using technology.
- ✔ Explore the system of complex numbers and find complex roots of quadratic equations.

### Patterns, Relations, and Algebra

- ✔ Explore matrices and their operations. Use matrices to solve systems of linear equations.
- ✔ Investigate recursive function notation.

### Geometry

- ✔ Apply properties of chords, tangents, and secants to solve problems.
- ✔ Use deduction to establish the validity of geometric conjectures and to prove theorems in Euclidean geometry.

### Measurement

- ✔ Explore the scientific use of different systems of measurement, e.g., centimeter-gram-second (CGS), Scientific International (SI).

### Data Analysis, Statistics, and Probability

- ✔ Explore designs of surveys, polls, and experiments to assess the validity of their results and to identify potential sources of bias; identify the types of conclusions that can be drawn.
- ✔ Describe the differences between the theoretical probability of simple events and the experimental outcome from simulations.

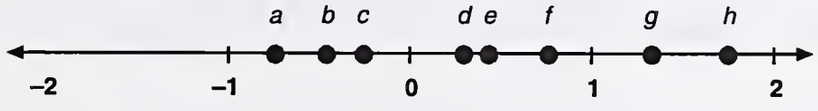
NOTE: The parentheses contain the code number(s) for the corresponding standard(s) in the single-subject courses.

## SELECTED PROBLEMS OR CLASSROOM ACTIVITIES FOR GRADES 9–10, ALGEBRA I, AND GEOMETRY

EXAMPLES

*Refers to standards 10.N.2, 10.N.3, and 10.N.4 (AI.N.2, AI.N.3, and AI.N.4)†*

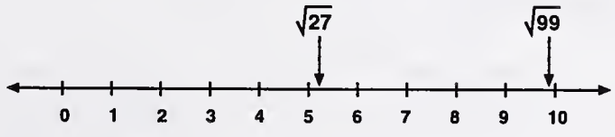
As high school students' understanding of numbers grows, they should learn to consider operations in general ways, rather than in only particular computations. The questions in the figure below call for reasoning about the properties of the numbers involved rather than for following procedures to arrive at exact answers. Such reasoning is important in judging the reasonableness of results. Although the questions can be approached by substituting approximate values for the numbers represented by *a* through *h,* teachers should encourage students to arrive at and justify their conclusions by thinking about properties of numbers. For example, to determine the point whose coordinate is closest to *ab,* a teacher might suggest considering the sign of *ab* and whether the magnitude of *ab* is greater or less than that of *b.* Likewise, students should be able to explain why, if *e* is positioned as given in the figure, the magnitude of $\sqrt{e}$ is greater than that of *e.* Listening to students explain their reasoning gives teachers insights into the sophistication of their arguments as well as their conceptual understanding.




Given the points with coordinates *a, b, c, d, e, f, g,* and *h* as shown, which point is closet to *ab*? To $|c|$? To $1/f$? To $\sqrt{e}$? To $\sqrt{h}$? Explain your reasoning.

*Refers to standard 10.N.3*

Locating square roots on the number line




$\sqrt{27}$ is a little more than 5 because $5^2 = 25$ and $6^2 = 36$.
$\sqrt{99}$ is a little less than 10 because $9^2 = 81$ and $10^2 = 100$.

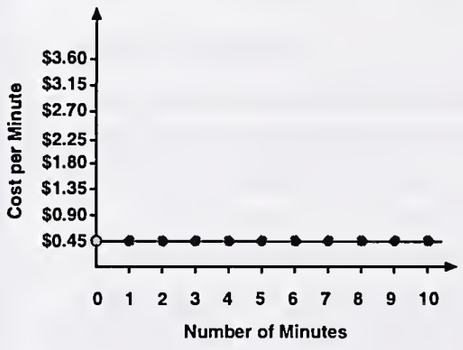
*Refers to standards 10.P.1 and 10.P.7 (AI.P.1 and AI.P.11)*

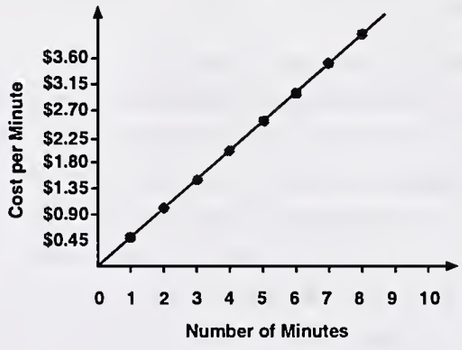
Research the changes in the number of cellular phones and personal computers in the United States between 1980 and 2000. First estimate, then use graphing calculators to decide whether the linear, quadratic, or exponential model is appropriate in each case. Compare growth rates and predict future changes in the use of each item. [The discussion may lead to topics in history and social studies related to growth and use of technology, including mathematical models to represent the changes.]

EXAMPLES

*Refers to standard 10.P.1 (AI.P.1)†*

These two graphs represent different relationships in a cellular telephone company's pricing scheme.




*Refers to standards 10.P.1 and 10.P.7 (AI.P.1 and AI.P.11)†*

Consider rectangles with a fixed area of 36 square units. The width *(W)* of the rectangles varies in relation to the length *(L)* according to the formula $W = 36/L$. Make a table showing the widths for all the possible whole-number lengths for these rectangles up to $L = 36$.

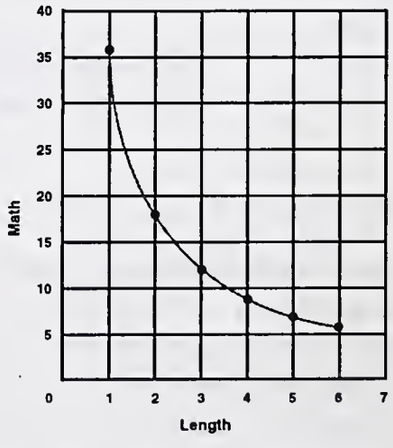
**Solution:**

| Length | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ... | 36 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Width | 36 | 18 | 12 | 9 | 7.2 | 6 | 5.14 | 4.5 | 4 | 3.6 | 3.27 | ... | 1 |

Look at the table and examine the pattern of the difference between consecutive entries for the length and the width. As the length increases by 1, the width decreases, but not at a constant rate. What do you expect the graph of the relationship between $L$ and $W$ to look like? Will it be a straight line?
Why or why not?

**Solution:**

**The graph is not a straight line because the rate of change is not constant. Instead the graph appears to be a curve that bends sharply downward and then becomes more level.**



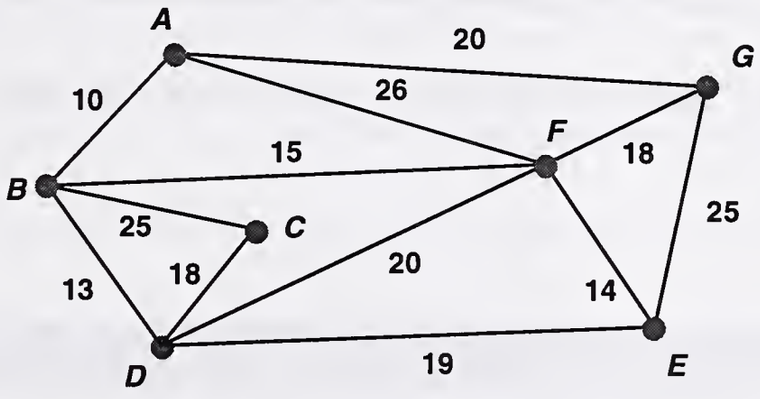
EXAMPLES

*Refers to standard 10.G.3 (G.G.6)*

Your shot put circle was washed out in a storm. There is only a portion left. You can redraw the circle if you know its center. Explain how you could use a geometric construction and the properties of circles to find the center of the original circle.

*Refers to standard 10.G.11 (G.G.17)†*

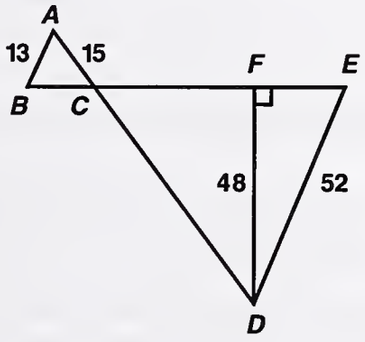
A vertex-edge graph depicting the lengths of roads between towns




*Refers to standards 10.G.4, 10.G.5, and 10.M.1 (G.G.5, G.G.7, and G.M.1)†*

A geometric problem requiring deduction and proof

In this figure, $\overline{AB} \parallel \overline{DE}$ and $\overline{DF} \perp \overline{CE}$. Determine the perimeter of $\triangle ABC$ and the perimeter of $\triangle CDE$. Explain completely how you found your answers and how you know they are correct.



EXAMPLES

*Refers to standards 10.D.1 and 10.D.2 (AI.D.1 and AI.D.2)*

Use an almanac to find the winning times for the women's 400-meter freestyle swim for the Olympics from 1924-1984.

1. On graph paper, using 1920 as the base year, plot (year, time).
2. Construct a best-fit line.
3. What is the slope and what does it mean?
4. Write the equation of the line. Use the line to predict what the times might has been if the Olympics had been held in 1940 and 1944.
5. Is it reasonable to use this line to predict the winning time for the 1988 Summer Games. Why or why not?
6. Look up the winning time for the 400-meter freestyle swim in the 1988 Summer Games and compare it to the time predicted by the best-fit line.

## Number Sense and Operations

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

### Grades 11–12 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

12.N.1 Define complex numbers (e.g., a + *bi*) and operations on them, in particular, addition, subtraction, multiplication, and division. Relate the system of complex numbers to the systems of real and rational numbers.

12.N.2 Simplify numerical expressions with powers and roots, including fractional and negative exponents.

## Patterns, Relations, and Algebra

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

### Grades 11–12 Learning Standards

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

12.P.1 Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative and recursive patterns such as Pascal's Triangle.

12.P.2 Identify arithmetic and geometric sequences and finite arithmetic and geometric series. Use the properties of such sequences and series to solve problems, including finding the general term and sum recursively and explicitly.

12.P.3 Demonstrate an understanding of the binomial theorem and use it in the solution of problems.

12.P.4 Demonstrate an understanding of the trigonometric, exponential, and logarithmic functions.

12.P.5 Perform operations on functions, including composition. Find inverses of functions.

12.P.6 Given algebraic, numeric and/or graphical representations, recognize functions as polynomial, rational, logarithmic, exponential, or trigonometric.

12.P.7 Find solutions to quadratic equations (with real coefficients and real or complex roots) and apply to the solutions of problems.

12.P.8 Solve a variety of equations and inequalities using algebraic, graphical, and numerical methods, including the quadratic formula; use technology where appropriate. Include polynomial, exponential, logarithmic, and trigonometric functions; expressions involving absolute values; trigonometric relations; and simple rational expressions.

12.P.9 Use matrices to solve systems of linear equations. Apply to the solution of everyday problems.

12.P.10 Use symbolic, numeric, and graphical methods to solve systems of equations and/or inequalities involving algebraic, exponential, and logarithmic expressions. Also use technology where appropriate. Describe the relationships among the methods.

12.P.11 Solve everyday problems that can be modeled using polynomial, rational, exponential, logarithmic, trigonometric, and step functions, absolute values, and square roots. Apply appropriate graphical, tabular, or symbolic methods to the solution. Include growth and decay; joint (e.g., $I = Prt$, $y = k(w_1 + w_2)$) and combined ($F = G(m_1m_2)/d^2$) variation, and periodic processes.

12.P.12 Relate the slope of a tangent line at a specific point on a curve to the instantaneous rate of change. Identify maximum and minimum values of functions in simple situations. Apply these concepts to the solution of problems.

12.P.13 Describe the translations and scale changes of a given function $f(x)$ resulting from substitutions for the various parameters a, b, c, and d in $y = af(b(x + c/b)) + d$. In particular, describe the effect of such changes on polynomial, rational, exponential, logarithmic, and trigonometric functions.

## GEOMETRY

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

### GRADES 11–12 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

12.G.1 Define the sine, cosine, and tangent of an acute angle. Apply to the solution of problems.

12.G.2 Derive and apply basic trigonometric identities (e.g., $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$) and the laws of sines and cosines.

12.G.3 Use the notion of vectors to solve problems. Describe addition of vectors and multiplication of a vector by a scalar, both symbolically and geometrically. Use vector methods to obtain geometric results.

12.G.4 Relate geometric and algebraic representations of lines, simple curves, and conic sections.

12.G.5 Apply properties of angles, parallel lines, arcs, radii, chords, tangents, and secants to solve problems.

## MEASUREMENT

Understand measurable attributes of objects and the units, systems, and processes of measurement

Apply appropriate techniques, tools, and formulas to determine measurements

### GRADES 11–12 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

12.M.1 Describe the relationship between degree and radian measures, and use radian measure in the solution of problems, in particular, problems involving angular velocity and acceleration.

12.M.2 Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense.

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

## GRADES 11–12 LEARNING STANDARDS

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

12.D.1 Design surveys and apply random sampling techniques to avoid bias in the data collection.

12.D.2 Select an appropriate graphical representation for a set of data and use appropriate statistics (e.g., quartile or percentile distribution) to communicate information about the data.

12.D.3 Apply regression results and curve fitting to make predictions from data.

12.D.4 Apply uniform, normal, and binomial distributions to the solutions of problems.

12.D.5 Describe a set of frequency distribution data by spread (i.e., variance and standard deviation), skewness, symmetry, number of modes, or other characteristics. Use these concepts in everyday applications.

12.D.6 Use combinatorics (e.g., "fundamental counting principle," permutations, and combinations) to solve problems, in particular, to compute probabilities of compound events. Use technology as appropriate.

12.D.7 Compare the results of simulations (e.g., random number tables, random functions, and area models) with predicted probabilities.

## Exploratory Concepts and Skills for Grades 11–12 and Single-Subject Courses

### Number Sense and Operations

- ✔ Investigate special topics in number theory, e.g., the use of prime numbers in cryptography.
- ✔ Use polar-coordinate representations of complex numbers (i.e., $a + bi = r(\cos\theta + i\sin\theta)$) and DeMoivre's theorem to multiply, take roots, and raise numbers to a power.
- ✔ Plot complex numbers using both rectangular and polar coordinate systems.

### Patterns, Relations, and Algebra

- ✔ Prove theorems using mathematical induction.
- ✔ Investigate parametrically defined curves and recursively defined functions, including applications to dynamic systems.

### Geometry

- ✔ Investigate and compare the axiomatic structures of Euclidean and non-Euclidean geometries.
- ✔ Explore the use of conic sections in engineering, design, and other applications.
- ✔ Investigate the notion of a fractal.
- ✔ Use graphs (networks) to investigate probabilistic processes and optimization problems.

### Data Analysis, Statistics, and Probability

- ✔ Use technology to perform linear, quadratic, and exponential regression on a set of data.
- ✔ Design surveys and apply random sampling techniques to avoid bias in the data collection.

NOTE: The parentheses contain the code number(s) for the corresponding standard(s) in the single-subject courses.

## SELECTED PROBLEMS OR CLASSROOM ACTIVITIES FOR GRADES 11–12, GEOMETRY, ALGEBRA II, AND PRECALCULUS

EXAMPLES

*Refers to standard 12.P.1 (AII.P.1)*

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
```

1. Construct the first 10 rows.
2. Identify different families or sets of numbers in the diagonals.
3. Relate the numbers in the triangle to the row numbers.
4. Examine sums of rows. Relate row sums to the row numbers.
5. For each row, form two sums by adding every other number. Compare sums within and between rows. Describe the patterns that emerge and why they occur.
6. Describe how the triangle is developed recursively.

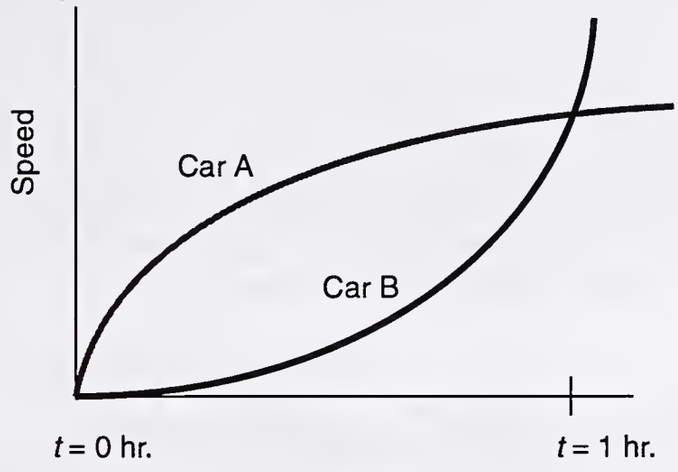
*Refers to standard 12.P.3 (AII.P.3) (TIMSS)*

Problem: Brighto soap powder is packed in cube-shaped cartons that measure 10 cm on each side. The company decides to increase the length of each side by 10%. How much does the volume increase?

Solution: $(10 + 1)^3 - 10^3 = (\cancel{10^3 \times 1^0} + 3 \times 10^2 \times 1^1 + 3 \times 10^1 \times 1^2 + 10^0 \times 1^3) - \cancel{10^3} = 331$, therefore the volume increases by $331 cm^3$.

EXAMPLES

*Refers to standards 12.P.1, 12.P.11, and 12.P.12 (AII.P.1, AII.P.11, and PC.P.9)†*




1. State the relationship between the position of car A and that of car B at $t = 1$ hr. Explain.
2. State the relationship between the velocity of car A and that of car B at $t = 1$ hr. Explain.
3. State the relationship between the acceleration of car A and that of car B at $t = 1$ hr. Explain.
4. How are the positions of the two cars related during the time interval between $t = 0.75$ hr. and $t = 1$ hr.? (That is, is one car pulling away from the other?) Explain.

*Refers to standards 12.P.8, 12.P.11, and 12.P.12 (AII.P.8, AII.P.11, and AII.P.12)*

A stone is thrown straight up into the air with initial velocity $v_0 = 10$ feet per second. If one neglects the effects of air resistance, after $t$ seconds the height of the stone is $h = v_0t - \frac{1}{2}gt^2$ (until the stone hits the ground), where $g \approx 32$ feet per second squared (the gravitational acceleration at the Earth's surface). What is the greatest height that the stone reaches, and when does it reach that height?

*Refers to standards 12.P.8, 12.P.11, and 12.G.2 (AII.P.8, AII.P.11, and AII.G.2)*

A stabilizing wire (guy wire) runs from the top of a 60 foot tower to a point 15 feet down the hill (measured on the slant) from the base of the tower. If the hill is inclined 11 degrees from the horizontal, how long does the wire need to be?

*Refers to standards 12.P.8, 12.P.11, and 12.G.1 (AII.P.8, AII.P.11, and AII.G.1)*

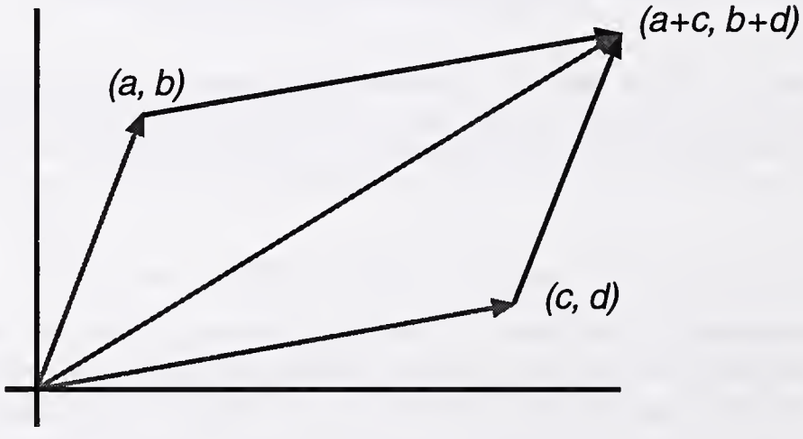
Students replicate the experiment in which Eratosthenes calculated the circumference of the earth and got a remarkably good answer. They locate some schools roughly due north or south of their school and connect with students in those schools through electronic mail. Students in each school agree that on a given day, at high noon, they will measure the shadow cast by a vertical stick on level ground. After sharing the measurements of the stick and the shadow, students use trigonometric ratios to determine the angle of the sun's rays. Using this information, along with the approximate distance between the schools, students use proportions to find an approximation of the earth's circumference. This example can be extended to sharing data with students from other states and countries.

*Refers to standards 12.P.8, 12.P.11, and 12.G.1 (AII.P.8, AII.P.11, and AII.G.1)*

How far from the horizontal must a sheet of plywood 4 feet wide be rotated to fit through a doorway 30 inches wide?

*Refers to standard 12.G.3 (G.G.18)†*

A simple vector sum.




*Refers to standard 12.M.1 (PC.M.1)*

In one hour, the minute hand on a clock moves through a complete circle, and the hour hand moves through 1/12 of a circle. Through how many radians do the minute and the hour hand move between 1:00 p.m. and 6:45 p.m. on the same day?

*Refers to standard 12.M.2 (PC.M.2)†*

High school students should be able to make reasonable estimates and sensible judgments about the precision and accuracy of the values they report. Teachers can help students understand that measurements of continuous quantities are always approximations. For example, suppose a situation calls for determining the mass of a bar of gold bullion in the shape of a rectangular prism whose length, width, and height are measured as 27.9 centimeters, 10.2 centimeters, and 6.4 centimeters, respectively. Knowing that the density is 19,300 kilograms per cubic meter, students might compute the mass as follows:

$$\text{Mass} = (\text{density}) \bullet (\text{volume})$$

$$= \left(19300 \frac{\text{kg}}{\text{m}^3}\right) \bullet \left(\frac{1}{10^6} \frac{\text{m}^3}{\text{cm}^3}\right) \bullet \left(1821.312\ \text{cm}^3\right)$$

$$= 35.1513216\ \text{kg}$$

The students need to understand that reporting the mass with this degree of precision would be misleading because it would suggest a degree of accuracy far greater than the actual accuracy of the measurement. Since the lengths of the edges are reported to the nearest tenth of a centimeter, the measurements are precise only to 0.05 centimeter. That is, the edges could actually have measures in the intervals $27.9 \pm 0.05$, $10.2 \pm 0.05$, and $6.4 \pm 0.05$. If students calculate the possible maximum and minimum mass, given these dimensions, they will see that at most one decimal place in accuracy is justified. As suggested by the example above, units should be reported along with numerical values in measurement computations.

*Refers to standard 12.D.6 (AII.D.2) (EDC, Inc)*

There are 9 points on a paper. No three are on the same line. How many different triangles can be drawn with vertices on these points?

*Refers to standard 12.D.6 (AII.D.2)*

There are eight McBride children, three girls and five boys. How many different ways are there of forming groups of McBride children containing at least two of the three girls?

*Refers to standard 12.D.6 (AII.D.2)*

Some services that involve electronic access require clients to choose a six-digit password. In an effort to increase security of the passwords, clients cannot use combinations that correspond to actual dates, nor can they use two identical digits in succession, nor passwords with one digit appearing three or more times. How many "secure" passwords are available?

*Refers to standard PC.P.1*

The goal of the Towers of Hanoi game pictured below is to move the tower on the left to either of the other two poles. At the end, the tower must look exactly as it does in the beginning, with the largest disk on the bottom and progressively smaller disks on top. You are allowed to move only one disk at a time. You are not allowed to place a larger disk on top of a smaller one. You must always move the top disk from a given tower.

You can play the game beginning with two or more disks in the tower. Could you use reasoning by mathematical induction to convince yourself that the object of the game is attainable no matter how many disks there are? Does induction suggest a particular recursive pattern for solving the problem as more disks are added? Can you determine the minimum number of moves as a function of the number of starting disks?

*Refers to standards 12.P.8 and 12.P.11 (AII.P.8 and AII.P.11)*

A solution's pH depends on the concentration of hydrogen ions per liter of the solution. The formula for determining a pH is

$$pH = \log \frac{1}{H^+}$$

where $H^+$ is the number of gram atoms of hydrogen ions per liter. The pH of neutral water is 7. Acidic solutions have a pH that is lower than 7, basic solutions have a pH that is higher than 7.

1. What is the pH of a solution with $4.231 \times 10^{-5}$ gram atoms of hydrogen ions per liter?
2. A certain juice has a pH of 3.9. Find the concentration of hydrogen ions of the juice.

# Learning Standards for Algebra I

NOTE: The parentheses at the end of a learning standard contain the code number for the corresponding standard in the two-year grade spans.

## NUMBER SENSE AND OPERATIONS

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AI.N.1 Identify and use the properties of operations on real numbers, including the associative, commutative, and distributive properties; the existence of the identity and inverse elements for addition and multiplication; the existence of $n^{th}$ roots of positive real numbers for any positive integer n; the inverse relationship between taking the $n^{th}$ root of and the $n^{th}$ power of a positive real number; and the density of the set of rational numbers in the set of real numbers. (10.N.1)

AI.N.2 Simplify numerical expressions, including those involving positive integer exponents or the absolute value, e.g., $3(2^4 - 1) = 45$, $4|3 - 5| + 6 = 14$; apply such simplifications in the solution of problems. (10.N.2)

AI.N.3 Find the approximate value for solutions to problems involving square roots and cube roots without the use of a calculator, e.g., $\sqrt{3^2 - 1} \approx 2.8$. (10.N.3)

AI.N.4 Use estimation to judge the reasonableness of results of computations and of solutions to problems involving real numbers. (10.N.4)

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AI.P.1 Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative, recursive (e.g., Fibonacci Numbers), linear, quadratic, and exponential functional relationships. (10.P.1)

AI.P.2 Use properties of the real number system to judge the validity of equations and inequalities, to prove or disprove statements, and to justify every step in a sequential argument.

AI.P.3 Demonstrate an understanding of relations and functions. Identify the domain, range, dependent, and independent variables of functions.

AI.P.4 Translate between different representations of functions and relations: graphs, equations, point sets, and tabular.

AI.P.5 Demonstrate an understanding of the relationship between various representations of a line. Determine a line's slope and x- and y-intercepts from its graph or from a linear equation that represents the line. Find a linear equation describing a line from a graph or a geometric description of the line, e.g., by using the "point-slope" or "slope y-intercept" formulas. Explain the significance of a positive, negative, zero, or undefined slope. (10.P.2)

AI.P.6 Find linear equations that represent lines either perpendicular or parallel to a given line and through a point, e.g., by using the "point-slope" form of the equation. (10.G.8)

AI.P.7 Add, subtract, and multiply polynomials. Divide polynomials by monomials. (10.P.3)

AI.P.8 Demonstrate facility in symbolic manipulation of polynomial and rational expressions by rearranging and collecting terms, factoring (e.g., $a^2 - b^2 = (a + b)(a - b)$, $x^2 + 10x + 21 = (x + 3)(x + 7)$, $5x^4 + 10x^3 - 5x^2 = 5x^2 (x^2 + 2x - 1)$), identifying and canceling common factors in rational expressions, and applying the properties of positive integer exponents. (10.P.4)

AI.P.9 Find solutions to quadratic equations (with real roots) by factoring, completing the square, or using the quadratic formula. Demonstrate an understanding of the equivalence of the methods. (10.P.5)

AI.P.10 Solve equations and inequalities including those involving absolute value of linear expressions (e.g., $|x - 2| > 5$) and apply to the solution of problems. (10.P.6)

## Patterns, Relations, and Algebra (continued)

AI.P.11 Solve everyday problems that can be modeled using linear, reciprocal, quadratic, or exponential functions. Apply appropriate tabular, graphical, or symbolic methods to the solution. Include compound interest, and direct and inverse variation problems. Use technology when appropriate. (10.P.7)

AI.P.12 Solve everyday problems that can be modeled using systems of linear equations or inequalities. Apply algebraic and graphical methods to the solution. Use technology when appropriate. Include mixture, rate, and work problems. (10.P.8)

## Data Analysis, Statistics, and Probability

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AI.D.1 Select, create, and interpret an appropriate graphical representation (e.g., scatterplot, table, stem-and-leaf plots, circle graph, line graph, and line plot) for a set of data and use appropriate statistics (e.g., mean, median, range, and mode) to communicate information about the data. Use these notions to compare different sets of data. (10.D.1)

AI.D.2 Approximate a line of best fit (trend line) given a set of data (e.g., scatterplot). Use technology when appropriate. (10.D.2)

AI.D.3 Describe and explain how the relative sizes of a sample and the population affect the validity of predictions from a set of data. (10.D.3)

# Learning Standards for Geometry

NOTE: The parentheses at the end of a learning standard contain the code number(s) for the corresponding standard(s) in the two-year grade spans.

## GEOMETRY

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

G.G.1 Recognize special types of polygons (e.g., isosceles triangles, parallelograms, and rhombuses). Apply properties of sides, diagonals, and angles in special polygons; identify their parts and special segments (e.g., altitudes, midsegments); determine interior angles for regular polygons. Draw and label sets of points such as line segments, rays, and circles. Detect symmetries of geometric figures.

G.G.2 Write simple proofs of theorems in geometric situations, such as theorems about congruent and similar figures, parallel or perpendicular lines. Distinguish between postulates and theorems. Use inductive and deductive reasoning, as well as proof by contradiction. Given a conditional statement, write its inverse, converse, and contrapositive.

G.G.3 Apply formulas for a rectangular coordinate system to prove theorems.

G.G.4 Draw congruent and similar figures using a compass, straightedge, protractor, or computer software. Make conjectures about methods of construction. Justify the conjectures by logical arguments. (10.G.2)

G.G.5 Apply congruence and similarity correspondences (e.g., $\triangle ABC \cong \triangle XYZ$) and properties of the figures to find missing parts of geometric figures, and provide logical justification. (10.G.4)

G.G.6 Apply properties of angles, parallel lines, arcs, radii, chords, tangents, and secants to solve problems.

G.G.7 Solve simple triangle problems using the triangle angle sum property, and/or the Pythagorean theorem. (10.G.5)

G.G.8 Use the properties of special triangles (e.g., isosceles, equilateral, 30°–60°–90°, 45°–45°–90°) to solve problems. (10.G.6)

## GEOMETRY (CONTINUED)

G.G.9 Define the sine, cosine, and tangent of an acute angle. Apply to the solution of problems.

G.G.10 Apply the triangle inequality and other inequalities associated with triangles (e.g., the longest side is opposite the greatest angle) to prove theorems and solve problems.

G.G.11 Demonstrate an understanding of the relationship between various representations of a line. Determine a line's slope and x- and y-intercepts from its graph or from a linear equation that represents the line. Find a linear equation describing a line from a graph or a geometric description of the line, e.g., by using the "point-slope" or "slope y-intercept" formulas. Explain the significance of a positive, negative, zero, or undefined slope. (10.P.2)

G.G.12 Using rectangular coordinates, calculate midpoints of segments, slopes of lines and segments, and distances between two points, and apply the results to the solutions of problems. (10.G.7)

G.G.13 Find linear equations that represent lines either perpendicular or parallel to a given line and through a point, e.g., by using the "point-slope" form of the equation. (10.G.8)

G.G.14 Demonstrate an understanding of the relationship between geometric and algebraic representations of circles.

G.G.15 Draw the results, and interpret transformations on figures in the coordinate plane, e.g., translations, reflections, rotations, scale factors, and the results of successive transformations. Apply transformations to the solution of problems. (10.G.9)

G.G.16 Demonstrate the ability to visualize solid objects and recognize their projections and cross sections. (10.G.10)

G.G.17 Use vertex-edge graphs to model and solve problems. (10.G.11)

G.G.18 Use the notion of vectors to solve problems. Describe addition of vectors and multiplication of a vector by a scalar, both symbolically and pictorially. Use vector methods to obtain geometric results. (12.G.3)

# MEASUREMENT

Understand measurable attributes of objects and the units, systems, and processes of measurement

Apply appropriate techniques, tools, and formulas to determine measurements

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

G.M.1 Calculate perimeter, circumference, and area of common geometric figures such as parallelograms, trapezoids, circles, and triangles. (10.M.1)

G.M.2 Given the formula, find the lateral area, surface area, and volume of prisms, pyramids, spheres, cylinders, and cones, e.g., find the volume of a sphere with a specified surface area. (10.M.2)

G.M.3 Relate changes in the measurement of one attribute of an object to changes in other attributes, e.g., how changing the radius or height of a cylinder affects its surface area or volume. (10.M.3)

G.M.4 Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. (10.M.4)

G.M.5 Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense. (12.M.2)

# Learning Standards for Algebra II

NOTE: The parentheses at the end of a learning standard contain the code number for the corresponding standard in the two-year grade spans.

## NUMBER SENSE AND OPERATIONS

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AII.N.1 Define complex numbers (e.g., a + b*i*) and operations on them, in particular, addition, subtraction, multiplication, and division. Relate the system of complex numbers to the systems of real and rational numbers. (12.N.1)

AII.N.2 Simplify numerical expressions with powers and roots, including fractional and negative exponents. (12.N.2)

## PATTERNS, RELATIONS, AND ALGEBRA

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AII.P.1 Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative and recursive patterns such as Pascal's Triangle. (12.P.1)

AII.P.2 Identify arithmetic and geometric sequences and finite arithmetic and geometric series. Use the properties of such sequences and series to solve problems, including finding the formula for the general term and the sum, recursively and explicitly. (12.P.2)

AII.P.3 Demonstrate an understanding of the binomial theorem and use it in the solution of problems. (12.P.3)

AII.P.4 Demonstrate an understanding of the exponential and logarithmic functions.

AII.P.5 Perform operations on functions, including composition. Find inverses of functions. (12.P.5)

AII.P.6 Given algebraic, numeric and/or graphical representations, recognize functions as polynomial, rational, logarithmic, or exponential. (12.P.6)

AII.P.7 Find solutions to quadratic equations (with real coefficients and real or complex roots) and apply to the solutions of problems. (12.P.7)

AII.P.8 Solve a variety of equations and inequalities using algebraic, graphical, and numerical methods, including the quadratic formula; use technology where appropriate. Include polynomial, exponential, and logarithmic functions; expressions involving the absolute values; and simple rational expressions. (12.P.8)

AII.P.9 Use matrices to solve systems of linear equations. Apply to the solution of everyday problems. (12.P.9)

AII.P.10 Use symbolic, numeric, and graphical methods to solve systems of equations and/or inequalities involving algebraic, exponential, and logarithmic expressions. Also use technology where appropriate. Describe the relationships among the methods. (12.P.10)

AII.P.11 Solve everyday problems that can be modeled using polynomial, rational, exponential, logarithmic, and step functions, absolute values and square roots. Apply appropriate graphical, tabular, or symbolic methods to the solution. Include growth and decay; logistic growth; joint (e.g., $I = Prt$, $y = k(w_1 + w_2)$), and combined ($F = G(m_1m_2)/d^2$) variation. (12.P.11)

AII.P.12 Identify maximum and minimum values of functions in simple situations. Apply to the solution of problems. (12.P.12)

AII.P.13 Describe the translations and scale changes of a given function $f(x)$ resulting from substitutions for the various parameters a, b, c, and d in $y = af\,(b(x + c/b)) + d$. In particular, describe the effect of such changes on polynomial, rational, exponential, and logarithmic functions. (12.P.13)

## Geometry

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AII.G.1 Define the sine, cosine, and tangent of an acute angle. Apply to the solution of problems. (12.G.1)

AII.G.2 Derive and apply basic trigonometric identities (e.g., $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$) and the laws of sines and cosines. (12.G.2)

AII.G.3 Relate geometric and algebraic representations of lines, simple curves, and conic sections. (12.G.4)

## Data Analysis, Statistics, and Probability

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

AII.D.1 Select an appropriate graphical representation for a set of data and use appropriate statistics (e.g., quartile or percentile distribution) to communicate information about the data. (12.D.2)

AII.D.2 Use combinatorics (e.g., "fundamental counting principle," permutations, and combinations) to solve problems, in particular, to compute probabilities of compound events. Use technology as appropriate. (12.D.6)

# Learning Standards for Precalculus

NOTE: The parentheses at the end of a learning standard contain the code number for the corresponding standard in the two-year grade spans.

## NUMBER SENSE AND OPERATIONS

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

Understand meanings of operations and how they relate to one another

Compute fluently and make reasonable estimates

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

PC.N.1 Plot complex numbers using both rectangular and polar coordinates systems. Represent complex numbers using polar coordinates, i.e., $a + bi = r(\cos\theta + i\sin\theta)$. Apply DeMoivre's theorem to multiply, take roots, and raise complex numbers to a power.

## PATTERNS, RELATIONS, AND ALGEBRA

Understand patterns, relations, and functions

Represent and analyze mathematical situations and structures using algebraic symbols

Use mathematical models to represent and understand quantitative relationships

Analyze change in various contexts

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

PC.P.1 Use mathematical induction to prove theorems and verify summation formulas, e.g., verify $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

PC.P.2 Relate the number of roots of a polynomial to its degree. Solve quadratic equations with complex coefficients.

## Patterns, Relations, and Algebra (continued)

PC.P.3 Demonstrate an understanding of the trigonometric functions (sine, cosine, tangent, cosecant, secant, and cotangent). Relate the functions to their geometric definitions.

PC.P.4 Explain the identity $\sin^2\theta + \cos^2\theta = 1$. Relate the identity to the Pythagorean theorem.

PC.P.5 Demonstrate an understanding of the formulas for the sine and cosine of the sum or the difference of two angles. Relate the formulas to DeMoivre's theorem and use them to prove other trigonometric identities. Apply to the solution of problems.

PC.P.6 Understand, predict, and interpret the effects of the parameters a, $\omega$, b, and c on the graph of $y = a\sin(\omega(x - b)) + c$; similarly for the cosine and tangent. Use to model periodic processes. (12.P.13)

PC.P.7 Translate between geometric, algebraic, and parametric representations of curves. Apply to the solution of problems.

PC.P.8 Identify and discuss features of conic sections: axes, foci, asymptotes, and tangents. Convert between different algebraic representations of conic sections.

PC.P.9 Relate the slope of a tangent line at a specific point on a curve to the instantaneous rate of change. Explain the significance of a horizontal tangent line. Apply these concepts to the solution of problems.

## Geometry

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Apply transformations and use symmetry to analyze mathematical situations

Use visualization, spatial reasoning, and geometric modeling to solve problems

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

PC.G.1 Demonstrate an understanding of the laws of sines and cosines. Use the laws to solve for the unknown sides or angles in triangles. Determine the area of a triangle given the length of two adjacent sides and the measure of the included angle. (12.G.2)

PC.G.2 Use the notion of vectors to solve problems. Describe addition of vectors, multiplication of a vector by a scalar, and the dot product of two vectors, both symbolically and geometrically. Use vector methods to obtain geometric results. (12.G.3)

PC.G.3 Apply properties of angles, parallel lines, arcs, radii, chords, tangents, and secants to solve problems. (12.G.5)

## Measurement

Understand measurable attributes of objects and the units, systems, and processes of measurement

Apply appropriate techniques, tools, and formulas to determine measurements

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

PC.M.1 Describe the relationship between degree and radian measures, and use radian measure in the solution of problems, in particular problems involving angular velocity and acceleration. (12.M.1)

PC.M.2 Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense. (12.M.2)

## Data Analysis, Statistics, and Probability

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them

Select and use appropriate statistical methods to analyze data

Develop and evaluate inferences and predictions that are based on data

Understand and apply basic concepts of probability

*Students engage in problem solving, communicating, reasoning, connecting, and representing as they:*

PC.D.1 Design surveys and apply random sampling techniques to avoid bias in the data collection. (12.D.1)

PC.D.2 Apply regression results and curve fitting to make predictions from data. (12.D.3)

PC.D.3 Apply uniform, normal, and binomial distributions to the solutions of problems. (12.D.4)

PC.D.4 Describe a set of frequency distribution data by spread (variance and standard deviation), skewness, symmetry, number of modes, or other characteristics. Use these concepts in everyday applications. (12.D.5)

PC.D.5 Compare the results of simulations (e.g., random number tables, random functions, and area models) with predicted probabilities. (12.D.7)

# Appendices

# Learning Standards by Grade Span for Grades PreK–6

## GRADES PREK – K

### NUMBER SENSE AND OPERATIONS

- Count by ones to at least 20.
- Match quantities up to at least 10 with numerals and words.
- Identify positions of objects in sequences (e.g., first, second) up to fifth.
- Compare sets of up to at least 10 concrete objects using appropriate language (e.g., none, more than, fewer than, same number of, one more than), and order numbers.
- Understand the concepts of whole and half.
- Identify U.S. coins by name.
- Use objects and drawings to model and solve related addition and subtraction problems to ten.
- Estimate the number of objects in a group and verify results.

### PATTERNS, RELATIONS, AND ALGEBRA

- Identify the attributes of objects as a foundation for sorting and classifying, e.g., a red truck, a red block, and a red ball share the attribute of being red; a square block, a square cracker, and a square book share the attribute of being square shaped.
- Sort and classify objects by color, shape, size, number, and other properties.
- Identify, reproduce, describe, extend, and create color, rhythmic, shape, number, and letter repeating patterns with simple attributes, e.g., ABABAB....
- Count by fives and tens at least up to 50.

# Learning Standards by Grade Span for Grades PreK–6

## Grades PreK – K

### Geometry

- Name, describe, sort, and draw simple two-dimensional shapes.
- Describe attributes of two-dimensional shapes, e.g., number of sides, number of corners.
- Name and compare three-dimensional shapes.
- Identify positions of objects in space, and use appropriate language (e.g., beside, inside, next to, close to, above, below, apart) to describe and compare their relative positions.

### Measurement

- Recognize and compare the attributes of length, volume/capacity, weight, area, and time using appropriate language, e.g., longer, taller, shorter, same length; heavier, lighter, same weight; holds more, holds less, holds the same amount.
- Make and use estimates of measurements from everyday experiences.
- Use nonstandard units to measure length, area, weight, and capacity.

### Data Analysis, Statistics and Probability

- Collect, sort, organize, and draw conclusions about data using concrete objects, pictures, numbers, and graphs.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 1 – 2

### Number Sense and Operations

- Name and write (in numerals) whole numbers to 1000, identify the place values of the digits, and order the numbers.
- Identify and distinguish among multiple uses of numbers, including cardinal (to tell how many) and ordinal (to tell which one in an ordered list), and numbers as labels and as measurements.
- Identify and represent common fractions (½, ⅓, ¼) as parts of wholes, parts of groups, and numbers on the number line.
- Compare whole numbers using terms and symbols, e.g., less than, equal to, greater than (<, =, >).
- Identify odd and even numbers and determine whether a set of objects has an odd or even number of elements.
- Identify the value of all U.S. coins, and $1, $5, $10, and $20 bills. Find the value of a collection of coins and dollar bills and different ways to represent an amount of money up to $5. Use appropriate notation, e.g., 69¢, $1.35.
- Demonstrate an understanding of various meanings of addition and subtraction, e.g., addition as combination (plus, combined with, more); subtraction as comparison (how much less, how much more), equalizing (how many more are needed to make these equal), and separation (how much remaining).
- Understand and use the inverse relationship between addition and subtraction (e.g., 8 + 6 = 14 is equivalent to 14 – 6 = 8 and is also equivalent to 14 – 8 = 6) to solve problems and check solutions.
- Know addition facts (addends to ten) and related subtraction facts, and use them to solve problems.
- Demonstrate the ability to add and subtract three-digit numbers accurately and efficiently.
- Demonstrate in the classroom an understanding of and the ability to use the conventional algorithms for addition (two 3-digit numbers and three 2-digit numbers) and subtraction (two 3-digit numbers).
- Estimate, calculate, and solve problems involving addition and subtraction of two-digit numbers. Describe differences between estimates and actual calculations.

### Patterns, Relations, and Algebra

- Identify, reproduce, describe, extend, and create simple rhythmic, shape, size, number, color, and letter repeating patterns.
- Identify different patterns on the hundreds chart.
- Describe and create addition and subtraction number patterns, e.g., 1, 4, 7, 10...; or 25, 23, 21....
- Skip count by twos, fives, and tens up to at least 50, starting at any number.
- Construct and solve open sentences that have variables, e.g., □ + 7 = 10.
- Write number sentences using +, –, <, = , and/or > to represent mathematical relationships in everyday situations.
- Describe functions related to trading, including coin trades and measurement trades, e.g., five pennies make one nickel, four cups make one quart, 11 nickels are worth more than 5 dimes.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 1 – 2

### Geometry

- Describe attributes and parts of two- and three-dimensional shapes, e.g., length of sides, and number of corners, edges, faces, and sides.
- Identify, describe, draw, and compare two-dimensional shapes, including both polygonal (up to six sides) and curved figures such as circles.
- Recognize congruent shapes.
- Identify shapes that have been rotated (turned), reflected (flipped), translated (slid), and enlarged. Describe direction of translations, e.g., left, right, up, down.
- Identify symmetry in two-dimensional shapes.
- Predict the results of putting shapes together and taking them apart.
- Relate geometric ideas to numbers, e.g., seeing rows in an array as a model of repeated addition.

### Measurement

- Identify parts of the day (e.g., morning, afternoon, evening), week, month, and calendar.
- Tell time at quarter-hour intervals on analog and digital clocks using a.m. and p.m.
- Compare the length, weight, area, and volume of two or more objects by using direct comparison.
- Measure and compare common objects using metric and English units of length measurement, e.g., centimeter, inch.
- Select and correctly use the appropriate measurement tools, e.g., ruler, balance scale, thermometer.
- Make and use estimates of measurement, including time, volume, weight, and area.

### Data Analysis, Statistics and Probability

- Use interviews, surveys, and observations to gather data about themselves and their surroundings.
- Organize, classify, represent, and interpret data using tallies, charts, tables, bar graphs, pictographs, and Venn diagrams; interpret the representations.
- Formulate inferences (draw conclusions) and make educated guesses (conjectures) about a situation based on information gained from data.
- Decide which outcomes of experiments are most likely.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 3 – 4

### Number Sense and Operations

- Exhibit an understanding of the base ten number system by reading, modeling, writing, and interpreting whole numbers to at least 100,000; demonstrating an understanding of the values of the digits; and comparing and ordering the numbers.
- Represent, order, and compare large numbers (to at least 100,000) using various forms, including expanded notation, e.g., $853 = 8 \times 100 + 5 \times 10 + 3$.
- Demonstrate an understanding of fractions as parts of unit wholes, as parts of a collection, and as locations on the number line.
- Select, use, and explain models to relate common fractions and mixed numbers (½, ⅓, ¼, ⅕, ⅙, ⅛, 1/10, 1/12, and 1½), find equivalent fractions, mixed numbers, and decimals, and order fractions.
- Identify and generate equivalent forms of common decimals and fractions less than one whole (halves, quarters, fifths, and tenths).
- Exhibit an understanding of the base ten number system by reading, naming, and writing decimals between 0 and 1 up to the hundredths.
- Recognize classes (in particular, odds, evens; factors or multiples of a given number; and squares) to which a number may belong, and identify the numbers in those classes. Use these in the solution of problems.
- Select, use, and explain various meanings and models of multiplication and division of whole numbers. Understand and use the inverse relationship between the two operations.
- Select, use, and explain the commutative, associative, and identity properties of operations on whole numbers in problem situations, e.g., $37 \times 46 = 46 \times 37$, $(5 \times 7) \times 2 = 5 \times (7 \times 2)$.
- Select and use appropriate operations (addition, subtraction, multiplication, division) to solve problems, including those involving money.

### Patterns, Relations, and Algebra

- Create, describe, extend and explain symbolic (geometric) and numeric patterns, including multiplication patterns like 3, 30, 300, 3000, . . .
- Use symbol and letter variables (e.g., △, x) to represent unknowns or quantities that vary in expressions and in equations or inequalities (mathematical sentences that use =, <, >).
- Determine values of variables in simple equations, e.g., $4106 - \nabla = 37$, $5 = ○ + 3$, and $□ - ○ = 3$.
- Use pictures, models, tables, charts, graphs, words, number sentences, and mathematical notations to interpret mathematical relationships.
- Solve problems involving proportional relationships, including unit pricing (e.g., four apples cost 80¢, so one apple costs 20¢) and map interpretation (e.g., one inch represents five miles, so two inches represent ten miles).
- Determine how change in one variable relates to a change in a second variable, e.g., input-output tables.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 3 – 4

### Geometry

- Compare and analyze attributes and other features (e.g., number of sides, faces, corners, right angles, diagonals, and symmetry) of two- and three-dimensional geometric shapes.
- Describe, model, draw, compare, and classify two- and three-dimensional shapes, e.g., circles, polygons—especially triangles and quadrilaterals—cubes, spheres, and pyramids.
- Recognize similar figures.
- Identify angles as acute, right, or obtuse.
- Describe and draw intersecting, parallel, and perpendicular lines.
- Using ordered pairs of numbers and/or letters, graph, locate, identify points, and describe paths (first quadrant).
- Describe and apply techniques such as reflections (flips), rotations (turns), and translations (slides) for determining if two shapes are congruent.
- Identify and describe line symmetry in two-dimensional shapes.
- Predict and validate the results of partitioning, folding, and combining two- and three-dimensional shapes.

### Measurement

- Demonstrate an understanding of such attributes as length, area, weight, and volume, and select the appropriate type of unit for measuring each attribute.
- Carry out simple unit conversions within a system of measurement, e.g., hours to minutes, cents to dollars, yards to feet or inches, etc.
- Identify time to the minute on analog and digital clocks using a.m. and p.m. Compute elapsed time using a clock (e.g., hours and minutes since . . .) and using a calendar (e.g., days since . . . ).
- Estimate and find area and perimeter of a rectangle, triangle, or irregular shape using diagrams, models, and grids or by measuring.
- Identify and use appropriate metric and English units and tools (e.g., ruler, angle ruler, graduated cylinder, thermometer) to estimate, measure, and solve problems involving length, area, volume, weight, time, angle size, and temperature.

### Data Analysis, Statistics and Probability

- Collect and organize data using observations, measurements, surveys, or experiments, and identify appropriate ways to display the data.
- Match representations of a data set such as lists, tables, or graphs (including circle graphs) with the actual set of data.
- Construct, draw conclusions, and make predictions from various representations of data sets, including tables, bar graphs, pictographs, line graphs, line plots, and tallies.
- Represent the possible outcomes for a simple probability situation, e.g., the probability of drawing a red marble from a bag containing three red marbles and four green marbles.
- List and count the number of possible combinations of objects from three sets, e.g., how many different outfits can one make from a set of three shirts, a set of two skirts, and a set of two hats?
- Classify outcomes as certain, likely, unlikely, or impossible by designating and conducting experiments using concrete objects such as counters, number cubes, spinners, or coins.

## Grades 3 – 4 (continued)

### Number Sense and Operations

- Know multiplication facts through 12 × 12 and related division facts. Use these facts to solve related multiplication problems and compute related problems, e.g., 3 × 5 is related to 30 × 50, 300 × 5, and 30 × 500.
- *Add and subtract (up to five-digit numbers) and multiply (up to three digits by two digits) accurately and efficiently.
- Divide up to a three-digit whole number with a single-digit divisor (with or without remainders) accurately and efficiently. Interpret any remainders.
- Demonstrate in the classroom an understanding of and the ability to use the conventional algorithms for addition and subtraction (up to five-digit numbers), and multiplication (up to three digits by two digits).
- Demonstrate in the classroom an understanding of and the ability to use the conventional algorithm for division of up to a three-digit whole number with a single-digit divisor (with or without remainders).
- Round whole numbers through 100,000 to the nearest 10, 100, 1000, 10,000, and 100,000.
- Select and use a variety of strategies (e.g., front-end, rounding, and regrouping) to estimate quantities, measures, and the results of whole-number computations up to three-digit whole numbers and amounts of money to $1000, and to judge the reasonableness of the answer.
- Use concrete objects and visual models to add and subtract common fractions.

### Patterns, Relations, and Algebra

* Although this standard is appropriate as stated for this grade span, the state assessment program at the 3–4 grade span will test multiplication of only up to two digits by two digits at the present time.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 3 – 4

| Geometry | Measurement | Data Analysis, Statistics and Probability |
|---|---|---|

# Learning Standards by Grade Span for Grades PreK–6

## Grades 5 – 6

### Number Sense and Operations

- Demonstrate an understanding of positive integer exponents, in particular, when used in powers of ten, e.g., $10^2$, $10^5$.
- Demonstrate an understanding of place value to billions and thousandths.
- Represent and compare very large (billions) and very small (thousandths) positive numbers in various forms such as expanded notation without exponents, e.g., $9724 = 9 \times 1000 + 7 \times 100 + 2 \times 10 + 4$.
- Demonstrate an understanding of fractions as a ratio of whole numbers, as parts of unit wholes, as parts of a collection, and as locations on the number line.
- Identify and determine common equivalent fractions, mixed numbers, decimals, and percents.
- Find and position integers, fractions, mixed numbers, and decimals (both positive and negative), on the number line.
- Compare and order integers (including negative integers), and positive fractions, mixed numbers, decimals, and percents.
- Apply number theory concepts—including prime and composite numbers, prime factorization, greatest common factor, least common multiple, and divisibility rules for 2, 3, 4, 5, 6, 9, and 10—to the solution of problems.
- Select and use an appropriate operation(s) to solve problems involving addition, subtraction, multiplication, division, and positive integer exponents with whole numbers, and with positive fractions, mixed numbers, decimals, and percents.
- Use the number line to model addition and subtraction of integers, with the exception of subtracting negative integers.
- Apply the Order of Operations for expressions involving addition, subtraction, multiplication, and division with grouping symbols (+, –, ×, ÷).
- Demonstrate an understanding of the inverse relationships of addition and subtraction, and use that understanding to simplify computation and solve problems.

### Patterns, Relations, and Algebra

- Analyze and determine the rules for extending symbolic, arithmetic, and geometric patterns and progressions, e.g., ABBCCC; 1, 5, 9, 13 . . . ; 3, 9, 27, . . .
- Replace variables with given values and evaluate/simplify, e.g., $2(\bigcirc) + 3$ when $\bigcirc = 4$.
- Use the property of equality to solve problems, e.g., if $\square + 7 = 13$, then $\square = 13 - 7$, therefore $\square = 6$; if $3 \times \square = 15$, then $\frac{1}{3} \times 3 \times \square = \frac{1}{3} \times 15$, therefore $\square = 5$.
- Represent real situations and mathematical relationships with concrete models, tables, graphs, and rules in words and with symbols, e.g., input-output tables.
- Solve linear equations using concrete models, tables, graphs, and paper-pencil methods.
- Produce and interpret graphs that represent the relationship between two variables in everyday situations.
- Identify and describe relationships between two variables with a constant rate of change. Contrast these with relationships where the rate of change is not constant.

# Learning Standards by Grade Span for Grades PreK–6

## Grades 5 – 6

### Geometry

- Identify polygons based on their properties, including types of interior angles, perpendicular or parallel sides, and congruence of sides, e.g., squares, rectangles, rhombuses, parallelograms, trapezoids, and isosceles, equilateral, and right triangles.
- Identify three-dimensional shapes (e.g., cubes, prisms, spheres, cones, and pyramids) based on their properties such as edges and faces.
- Identify relationships among points, lines, and planes (e.g., intersecting, parallel, perpendicular).
- *Graph points and identify coordinates of points on the Cartesian coordinate plane (all four quadrants).
- Find the distance between two points on horizontal or vertical number lines.
- Predict, describe and perform transformations on two-dimensional shapes, e.g., translations, rotations, and reflections.
- Identify types of symmetry, including line and rotational.

* Although this standard is important and appropriate for this grade span, it will not be included in the state assessment program at the 5–6 grade span at the present time.

### Measurement

- Apply the concepts of perimeter and area to the solution of problems. Apply formulas where appropriate.
- Identify, measure, describe, classify, and construct various angles, triangles, and quadrilaterals.
- Solve problems involving proportional relationships and units of measurement, e.g., same system unit conversions, scale models, maps, and speed.
- Find areas of triangles and parallelograms. Recognize that shapes with the same number of sides but different appearances can have the same area. Develop strategies to find the area of more complex shapes.
- Identify, measure, and describe circles and the relationships of the radius, diameter, circumference, and area (e.g., $d = 2r$, $\pi = C/d$), and use the concepts to solve problems.
- Find volumes and surface areas of rectangular prisms.
- Find the sum of the angles in simple polygons (up to eight sides) with and without measuring the angles.

### Data Analysis, Statistics and Probability

- Describe and compare data sets using the concepts of median, mean, mode, maximum and minimum, and range.
- Construct and interpret stem-and-leaf plots, line plots, and circle graphs.
- Use tree diagrams and other models (e.g., lists and tables) to represent possible or actual outcomes of trials. Analyze the outcomes.
- Predict the probability of outcomes of simple experiments (e.g., tossing a coin, rolling a die) and test the predictions. Use appropriate ratios between 0 and 1 to represent the probability of the outcome and associate the probability with the likelihood of the event.

# Learning Standards by Grade Span for Grades PreK–6

## GRADES 5 – 6 (CONTINUED)

### NUMBER SENSE AND OPERATIONS

- Accurately and efficiently add, subtract, multiply, and divide (with double-digit divisors) whole numbers and positive decimals.
- Accurately and efficiently add, subtract, multiply, and divide positive fractions and mixed numbers. Simplify fractions.
- Add and subtract integers, with the exception of subtracting negative integers.
- Estimate results of computations with whole numbers, and with positive fractions, mixed numbers, decimals, and percents. Describe reasonableness of estimates.

### PATTERNS, RELATIONS, AND ALGEBRA

# Learning Standards by Grade Span for Grades PreK–6

## GRADES 5 – 6 (CONTINUED)

| GEOMETRY | MEASUREMENT | DATA ANALYSIS, STATISTICS AND PROBABILITY |
|---|---|---|
| • Determine if two shapes are congruent by measuring sides or a combination of sides and angles, as necessary; or by motions or series of motions, e.g., translations, rotations, and reflections.<br>• Match three-dimensional objects and their two-dimensional representations, e.g., nets, projections, and perspective drawings. | | |

# Criteria for Evaluating Instructional Materials and Programs in Mathematics

| | Strongly Agree | Agree | Cannot Judge | Disagree | Strongly Disagree |
|---|---|---|---|---|---|
| **I. Mathematical Content** | | | | | |
| Reflects the learning standards in the Mathematics Curriculum Framework | | | | | |
| Is mathematically accurate | | | | | |
| **II Features** | | | | | |
| Provide descriptions of the achievements of historically important mathematicians | | | | | |
| Contain illustrations of contemporary children and adults that reflect the diversity of our society | | | | | |
| Include clear instructions on using tools, equipment, and materials | | | | | |
| Include a master source of materials and resources | | | | | |
| Provide student texts, booklets, or printed material and accompanying teacher manuals | | | | | |
| Provide coherent units that build conceptual understanding | | | | | |
| Provide for in-depth investigations of major mathematical concepts | | | | | |
| Incorporate applications of mathematics | | | | | |
| Highlight connections within mathematics and with the natural and social sciences where relevant | | | | | |
| **III Learning Activities** | | | | | |
| Involve students in active learning, inquiry, and problem solving | | | | | |
| Involve the use of manipulatives to explore, model, and analyze | | | | | |
| Clarify appropriate use of instructional technology such as calculators and computers | | | | | |
| Show how instructional technology can help students visualize complex concepts, analyze and refine information, and communicate solutions | | | | | |
| Provide multiple ways for students to explore concepts and communicate ideas and solutions | | | | | |
| Are developmentally appropriate and provide for different abilities and learning paces | | | | | |
| Encourage discussion and reflection | | | | | |

# Criteria for Evaluating Instructional Materials and Programs in Mathematics (Continued)

| | Strongly Agree | Agree | Cannot Judge | Disagree | Strongly Disagree |
|---|---|---|---|---|---|
| Draw on a variety of resources (e.g., trade manuals, measuring tools, other tools and machines, manipulatives, and the internet) | | | | | |
| Show students how to use standard algorithms for basic arithmetic operations | | | | | |
| **IV Teacher Support Materials** | | | | | |
| Provide a clear conceptual framework for the concepts and skills taught | | | | | |
| Offer ideas for involving parents and community and keeping them informed about the programs | | | | | |
| Give suggestions for a variety of pedagogical strategies, such as open-ended questioning, direct instruction, practice, discussion, and cooperative learning | | | | | |
| Reference resource materials, such as appropriate videos, file clips, reference books, software, video laser disk, long-distance learning, CD-ROM, and electronic bulletin boards | | | | | |
| Suggest how to adapt materials for students with differing levels of achievement | | | | | |
| Suggest enrichment and skill reinforcement activities for extended learning | | | | | |
| Include suggestions for a variety of assessment approaches such as portfolios, journals, projects, and informal and formal tests | | | | | |
| **V Student Assessment Materials** | | | | | |
| Are free of inappropriate or derogatory material | | | | | |
| Contain a balance among activities that assess conceptual understanding, procedural skill, and problem solving ability | | | | | |
| Occur throughout the unit, not just at the end | | | | | |
| Incorporate multiple forms of assessment, such as oral presentations, written reports, teacher observations, performance assessments, quizzes, and pre- and post-tests | | | | | |
| Focus on the acquisition of skills and concepts as well as on the learning process | | | | | |
| **VI Program Development and Implementation** | | | | | |
| Have field test data showing positive effects on student learning | | | | | |
| Are adaptable to local curriculum and/or school | | | | | |
| Offer training and long-term follow up for teachers | | | | | |

# Glossary

**Glossary Sources**

(H) Harcourt Animated Math Glossary. Referenced February 29–March 2, 2000. <www.hbschool.com/glossary/math/glossary8.html>

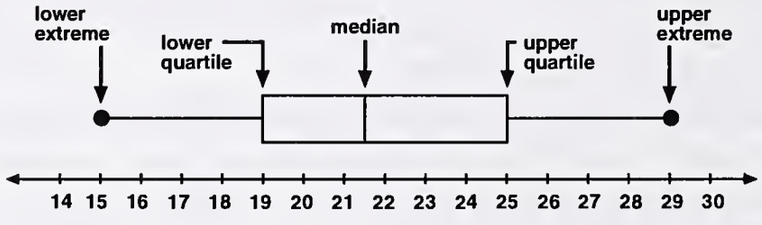
(W) Merriam-Webster's Collegiate Dictionary. Referenced February 29–March 2, 2000. <m-w.com/dictionary.htm>

**Algorithm** A finite set of steps for completing a procedure, e.g., long division. (H)

**Analog** Having to do with data represented by continuous variables, e.g., a clock with hour, minute, and second hands. (W)

**Analytic geometry** The branch of mathematics that uses functions and relations to study geometric phenomena, e.g., the description of ellipses and other conic sections in the coordinate plane by quadratic equations.

**Box-and-whisker plot** A method for displaying the median, quartiles, and extremes of a set of data, using the number line. (H)




**Calculus** The mathematics of change and motion. The main concepts of calculus are limits, instantaneous rates of change, and areas enclosed by curves.

**Capacity** The maximum amount or number that can be contained or accommodated, e.g., a jug with a one-gallon capacity; the auditorium was filled to capacity.

**Capture-recapture experiment** A type of experiment used to estimate a population. On the first day a certain number, M, of the population, N, is captured and tagged or marked. On the second day another sample, n, of the population is captured including the recapture of some R of those already tagged. The population can be estimated using the formula $N = {}^{Mn}/_{R}$.

**Cardinal number** A number (as 1, 5, 15) that is used in simple counting and that indicates how many elements there are in a set.

**Cartesian plane or Euclidean plane** A coordinate plane with perpendicular coordinate axes.

**Closure property** A set of numbers, such as the integers, is closed under a particular operation if performing the operation on numbers in the set results in another number in that set. For example, the set of non-zero integers is closed under multiplication, but is not closed under division: the product of two non-zero integers is again a non-zero integer, but their quotient need not be an integer.

# Glossary

**Complex number** A number that can be written in the form a + bi where a and b are real numbers and $i = \sqrt{-1}$

**Convenience sample** The collection of data from readily available sources, without emphasis on methodological rigor.

**Coordinate plane** A plane in which two coordinate axes are specified, i.e., two intersecting directed straight lines, usually perpendicular to each other, and usually called the x-axis and y-axis. Every point in a coordinate plane can be described uniquely by an ordered pair of numbers, the coordinates of the point with respect to the coordinate axes.

**Counting number** A number used in counting objects, i.e., a number from the set 1, 2, 3, 4, 5, . . .

**Decimal number** Any real number expressed in base 10 notation, such as 2.673.

**Deductive reasoning** A type of reasoning wherein the conclusion about particulars follows necessarily from general or universal premises. (W)

**Digit** a) Any of the Arabic numerals 1 to 9 and usually the symbol 0; b) One of the elements that combine to form numbers in a system other than the decimal system.

**Digital** Having to do with data that is represented in the form of numerical digits; providing a readout in numerical digits, e.g., a digital watch.

**Dilation** A type of transformation of the plane that fixes a point *C* (the *center* of the dilation) and maps any other point *P* to the point *P'* characterized as follows: the segment CP' has the same direction as the segment *CP,* and has length *k* times the length of the segment CP, where k is a positive constant (the *scale factor* of the dilation).

**Discrete mathematics** The branch of mathematics that includes combinatorics, recursion, Boolean algebra, set theory, and graph theory.

**Dot product** For vectors $A = \langle x_a, y_a \rangle$ and $B = \langle x_b, y_b \rangle$, the dot product $A \bullet B = (x_a)(x_b) + (y_a)(y_b)$.

**Expanded notation** A way of representing a number that shows the sum of each digit multiplied by the appropriate positive power of ten and the units digit, e.g., 3451 as $3 \times 1000 + 4 \times 100 + 5 \times 10 + 1$ or as $3 \times 10^3 + 4 \times 10^2 + 5 \times 10 + 1$.

**Exponent** The number that indicates how many times the base is used as a factor, e.g., in $4^3 = 4 \times 4 \times 4 = 64$, the exponent is 3, indicating that 4 is repeated as a factor three times.

**Fibonacci numbers** The sequence of numbers beginning with 1, 1, in which each number that follows is the sum of the previous two numbers, i.e., 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 . . .

**Function** A mathematical relation that associates each object in a set with exactly one value.

**Fundamental Counting Principle** If event M can occur in m ways, and after it has occurred, event N can occur in n ways, then event M followed by event N can occur m•n ways.

# Glossary

**Fractal** A curve or shape so irregular that its dimension (according to the technical definition of dimension) is a *fraction*, rather than an integer. Many interesting examples of fractals have the property of being *self-similar,* in the sense that portions of the fractal are similar in shape to magnified parts of itself at arbitrarily high rates of magnification.

**Geometric or figural pattern** A sequence of symbols or geometric figures.

**Geometric sequence (progression)** An ordered list of numbers that has a common ratio between consecutive terms, e.g., 2, 6, 18, 54 . . . (H)

**Inductive reasoning** a) The type of reasoning that uses inference to reach a generalized conclusion from particular instances; b) In mathematics, demonstration of the validity of a law concerning all the positive integers by proving that it holds for the integer 1 and that if it holds for an arbitrarily chosen positive integer $k$ it must hold for the integer $k+1$; also called *mathematical induction.* (W)

**Integer** A number that is either a whole number or the negative of a whole number.

**Irrational number** A number that cannot be expressed as a quotient of two integers, e.g., $\sqrt{2}$. It can be shown that a number is irrational if and only if it cannot be written as a repeating or terminating decimal.

**Iterative pattern or sequence** A sequence or pattern formed by repeating the same procedure. For example, the Fibonacci sequence.

**Line graph** A set of data points on an X-Y grid, possibly with consecutive points connected by line segments.

**Line of best fit** A straight line drawn through or near to as many data points as possible on a scatterplot.

**Line plot** A number line with dots or other marks above it to show the number of times an event occurs. (H)

**Linear equation** Any equation that can be written in the form $Ax + By + C = 0$ where A and B cannot both be 0. The graph of such an equation is a line.

**Linear programming** A mathematical method of solving practical problems (as the allocation of resources) by means of linear functions where the variables involved are subject to constraints. (W)

**Matrix, pl. matrices** A rectangular array of numbers or variables.

**Measure of central tendency** Either the mean, median, or mode of a distribution, i.e., one of the possible notions of an average.

**Network** a) A figure consisting of vertices and edges that shows how objects are connected, b) A collection of points (vertices), with certain connections (edges) between them.

**Numeral** A symbol or mark used to represent a number.

**Numeric pattern** A pattern composed of numerals.

# Glossary

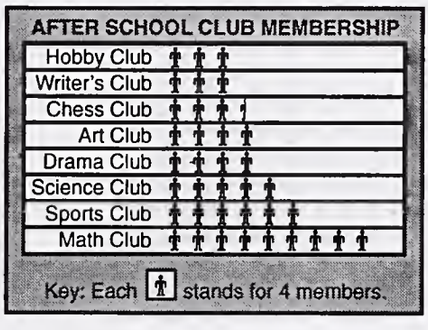
**Order of Operations** 1. Do all of the operations inside parentheses, and/or above and below a fraction bar in the proper order, 2. Find the value of any powers or roots, 3. Multiply and divide from left to right, 4. Add and subtract from left to right. (H)

**Ordinal number** A number designating the place (as first, second, or third) occupied by an item in an ordered sequence. (W)

**Partition** A process of dividing an object into parts.

**Pascal's triangle** A triangular arrangement of numbers in which each row starts and ends with 1, and each other number is the sum of the two numbers above it. (H)

**Pictogram (pictograph)** A graph that uses pictures to show and compare information. (H)




**Probability** A number between zero and one that describes the likelihood that a given event will take place. For example, the probability of throwing a six with a single throw of one die is ⅙, and the probability of throwing two sixes with a single throw of two dice is 1/36.

**Proof** A method of constructing a valid argument, using deductive reasoning.

**Proportion** An equation that states that two ratios are equivalent, e.g., 4/8 = ½ or 4 : 8 = 1 : 2.

**Pythagorean theorem** For any right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

**Random sampling** A smaller group of people or objects chosen from a larger group or population by a process giving equal chance of selection to all possible people or objects. (H)

**Random variable** A variable that is itself a function of the result of a statistical experiment in which each outcome has a definite probability of occurrence; also called *variate.* (W)

# Glossary

**Ratio** A comparison of two numbers or quantities, e.g., 4 to 7 or 4 : 7 or $\frac{4}{7}$.

**Rational number** A number that can be written as the ratio of an integer to a counting number; or more formally, a number that can be expressed as a ratio a/b where a and b are integers and $b \neq 0$, e.g., 0.5, $\frac{3}{5}$, -3, 8, $3\frac{9}{10}$.

**Real number** A number from the set of numbers consisting of all rational and all irrational numbers.

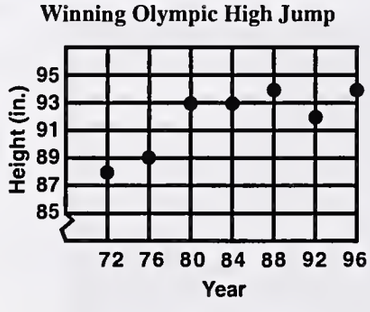
**Recursive pattern or sequence** A pattern or sequence wherein each successive term can be computed from some or all of the preceding terms by an algorithmic procedure.

**Reflection** A type of transformation that flips points about a line, called the *line of reflection.* Taken together, the image and the preimage have the line of reflection as a line of symmetry.

**Rotation** A type of transformation that turns a figure about a fixed point, called the *center of rotation.*

**Sample space** In probability, the set of all outcomes of a given experiment, e.g., the sample space for tossing two coins is (H,H), (H,T), (T,H), (T,T). (H)

**Scatter plot** Two sets of data plotted as ordered pairs in the coordinate plane. (H)




**Scientific notation** A widely used floating-point system in which numbers are expressed as products consisting of a number between 1 and 10 multiplied by an appropriate power of 10, e.g., $562 = 5.62 \times 10^2$. (W)

**Sequence, progression** A set of elements ordered so that they can be labeled with consecutive positive integers starting with 1, e.g., 1, 3, 9, 27, 81. In this sequence, 1 is the *first term,* 3 is the *second term*, 9 is the *third term,* and so on. (W)

**Transformation** A prescription, or rule, that sets up a one-to-one correspondence between the points in a geometric object (the *preimage*) and the points in another geometric object (the *image*). Reflections, rotations, translations, and dilations are particular examples of transformations.

**Translation** A type of transformation that moves every point by the same distance in the same direction, e.g., in a geographic map, moving a given distance due north.

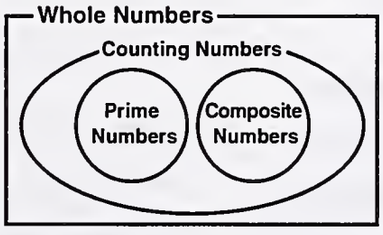
**Valid** a) Well-grounded or justifiable; being at once relevant and meaningful, e.g., a *valid* theory; b) Logically correct. (W)

# Glossary

**Variable** A letter used to represent one or more numbers in an expression, equation, inequality, or matrix. (H)

**Vector** A quantity that has magnitude and direction. A vector is typically represented by a directed line segment, whose length represents the magnitude and whose orientation in space represents the direction. (W)

**Venn diagram** A diagram that is used to show relationships between sets. (H)




**Vertex-edge graph** A graph consisting of points with connections in which the points are called vertices and the line segments are called edges.

**Whole number** A number that is either a counting number or zero.

# Internet Resources

## Curriculum and Assessment Resources

| | |
|---|---|
| K-12 Mathematics Curriculum Center | www.edc.org/mcc |
| Mathematics Archives K-12 Teaching Material | archives.math.utk.edu/k12.html |
| NCTM 2000, Principles and Standards for School Mathematics | standards.nctm.org/document/index.htm |
| Middle School Curriculum | showmecenter.missouri.edu |
| Curriculum Library Alignment and Sharing Project (CLASP) | www.massnetworks.org/clasp/clasp.html |
| Eisenhower National Clearinghouse for Mathematics and Science Education | www.enc.org |
| Massachusetts Department of Education | www.doe.mass.edu |
| PALMS Initiative | www.doe.mass.edu/palms |
| Mathematics Curriculum Frameworks | www.doe.mass.edu/frameworks |

## General Mathematics Education and Teacher Resources

| | |
|---|---|
| Mathematical Association of America (MAA) | www.maa.org |
| Mass Ed Online Learnet | med.mass.edu |
| Quality Educators for Minorities Network | qemnetwork.qem.org |
| Voices of Girls in Math, Science and Technology | www.ael.org/nsf |
| TERC | www.terc.edu |
| Math Forum | forum.swarthmore.edu |
| The Geometry Junkyard | www.ics.uci.edu/~eppstein/junkyard |
| National Council of Teachers of Mathematics | www.nctm.org |
| Math Teacher Link | www.mtl.math.uiuc.edu |
| Busy Teachers' Web Site | www.ceismc.gatech.edu/busyt/math.html |
| Global Schoolhouse | www.gsh.org |
| Ask ERIC Home Page | ericir.syr.edu |
| The Geometry Center | freeabel.geom.umn.edu |
| Elementary School Teachers' Place | forum.swarthmore.edu/teachers/elem |
| American Mathematical Society | www.ams.org |
| National Science Foundation | www.nsf.gov |

# Internet Resources

**International Sites**

| | |
|---|---|
| Japanese Math Challenge | japanese-online.com/math/index.htm |
| NRICH Online Math Club | www.nrich.maths.org.uk |
| Third International Mathematics and Science Study (TIMSS) | ustimss.msu.edu |

**Math Contests**

| | |
|---|---|
| The Centre for Education in Mathematics and Computing | math.uwaterloo.ca:80/~cemc |
| Australian Mathematics Trust | www.amt.canberra.edu.au |
| Math League | www.mathleague.com |

**History of Mathematics**

| | |
|---|---|
| Women in Math Project | Darkwing.uoregon.edu/wmnmath/ |
| History of Mathematics Home Page | aleph0.clarku.edu/~djoyce/mathhist/mathhist.html |
| Mac Tutor History of Math Archive | www-groups.dcs.st-and.ac.uk:80/~history |
| Mathematical Quotations | math.furman.edu/~mwoodard/ascquotd.html |

**Number Theory**

| | |
|---|---|
| Math Fun with Spirolaterals | www.corona.bell.k12.ca.us/teach/swa/sept.html |
| Fantastic Fractals/Discover Fractals | library.advanced.org/12740 |
| The Centre for Experimental and Constructive Mathematics | cecm.sfu.ca |

**Puzzles and Games**

| | |
|---|---|
| Puzzle Archives | alabanza.com/kabacoff/Inter-Links/puzzles.html |
| Brain Teasers | www.eduplace.com/math/brain/index.html |
| Internet Math Challenge-U Idaho | www.uidaho.edu/LS/Math/imc |
| Interactive Mathematics Miscellany and Puzzles | www.cut-the-knot.com |
| Number and Word Puzzles | www1.ptgi.com.au/users/puzzles |

**Mathematics Dictionaries**

| | |
|---|---|
| The Famous Dictionary of Mathematics | library.thinkquest.org/3087/dictiona.html |
| Harcourt Animated Math Glossary | www.hbschool.com/glossary/math |

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## Endnotes

[1] Massachusetts Department of Education, internal report, Course Progression and Placement Leading to Enrollment in Advanced Placement Calculus in the Twelfth Grade, March 2000.

[2] U.S. Department of Education, "Mathematics Equals Opportunity" (white paper prepared for U.S. Secretary of Education Richard W. Riley), October 20, 1997. <ed.gov/pubs/math>

[3] Ma, Liping, *Knowing and Teaching Elementary Mathematics*, Mahwah, New Jersey: Lawrence Erlbaum Associates, 1999.

[4] Milken, Lowell, *A Matter of Quality: A Strategy for Assuring the High Caliber of America's Teachers,* Santa Monica, California: Milken Family Foundation, 1999, p. 8. [drawn from Sanders and Rivers, see Selected Bibliography]

[5] Ibid, p. 16 [drawn from Ingersoll, see Selected Bibliography]

[6] Ma, p. 147.

[7] "U.S. fourth graders use calculators and computers in mathematics class more frequently than do students in most other TIMSS countries. Use of calculators in U.S. fourth-grade mathematics classes is about twice the international average . . . . In six of the seven nations that outscore the U.S. in mathematics, teachers of 85 percent or more of the students report that students never [or hardly ever] use calculators in class." National Center for Education Statistics, *Pursuing Excellence: A Study of U.S. Fourth-Grade Mathematics and Science Achievement in International Context,* chapter 2 "Contexts of Learning," accessed June 15, 2000, <nces.ed.gov/timss/report/97255-2a.html>.

[8] National Council of Teachers of Mathematics, *NCTM Standards 2000, Principles and Standards for School Mathematics,* April 2000.

[9] As the *Notices of the American Mathematical Society* state, standard algorithms "are more than just 'ways to get the answer'—that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra . . . . The division algorithm is also significant for later understanding of real numbers." American Mathematical Society NCTM 2000 Association Resource Group, *Notices of the American Mathematical Society, Second Report,* February 1998.

[10] National Council of Teachers of Mathematics, *NCTM Standards 2000, Principles and Standards for School Mathematics,* April 2000.

[11] Pratt Institute, *The Architectural Event,* accessed June 15, 2000, <acnet.pratt.edu/~arch543p/help/euclidean_geometry.html>.

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